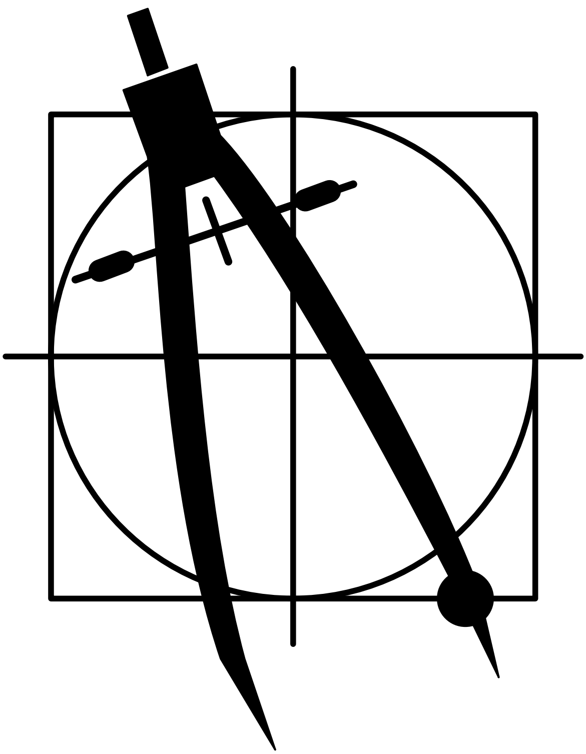
A Young Person's Guide to Competition Mathematics



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A YOUNG PERSON'S GUIDE TO COMPETITION MATHEMATICS

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Dedicated to the Math Team students of Lassiter High School.

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FIRST PREFACE

For years Lassiter Math Team students asked if they could host a math tournament, to which we replied that if they brought us a written test and ciphering questions we would send out invitations. Then in 2000, Lassiter High School student Garrett Webb showed up with a complete contest. He wrote the first Lassiter Invitational Mathematics Tournament and started a tradition that has lived on for at least 18 years. Students at Lassiter write and run this annual tournament that has hosted up to 500 students from over 50 schools. Garrett and Ethan Trewitt were the main contributors during the first 2 years, but writing the tournament is now an entire math team effort, with almost all members of the math team submitting potential questions.

Because of how the team works together to edit questions, it would be impossible to list all the students who have contributed questions for the Lassiter Invitational Tournaments. However, usually one or two students will take responsibility for actually creating and editing the tests, writing the solutions and running the competition. Some of these student leaders have been Garrett and Ethan, Matt Speer, Brooks Andrews, Jonathan Rodean, Andy Vesper, Adam Tart, Carine Davila, Michael Clark, Katie Vesper, Phillip Mote, Martin Copenhaven, Miles Dillon Edwards, Michael Wilson, Katie Crane, Andrew Couch, Brian Cohn, and Nicholas Lindell. We thank these students as well as all the other wonderful math team members from Lassiter High School.

And most of all, we are so thankful that Chuck Garner took the time to organize these questions in a logical order and write such wonderful explanations of the concepts presented. He really did fulfill Garrett's original desire that students would learn mathematics from the Lassiter Invitational as well as compete in a mathematics tournament. Thank you, Chuck for all your hard work.

Debbie Poss and Don Slater
November 2017

SECOND PREFACE

Debbie Poss and Don Slater are master teachers. They have nurtured such wonderful mathematical talent in their students over the years, as evidenced by the problems in this collection. All the problems in this book come from the annual Lassiter High School Inivtational Mathematics Tournament, held the first Saturday in December each year. The Lassiter Math Team students write the problems, and are edited by Debbie and Don, so they are ultimately responsible for the quality of the problems. However, the students they mentor do such a fantastic job coming up with challenging and intriguing problems, Don and Debbie are really responsible for the entire process.

The problems contained herein are from the 2003 through 2012 LHS tournaments. Each tournament offers a 30-problem written test, 10-problem individual ciphering, and a multi-part team round called a "power question." And each of these rounds are offered in both varsity and junior varsity divisions. The test is unique in that only the first 25 problems are multiple-choice; the last 5 problems on each test are free-response. Thus, with two 30-problem tests each year for 10 years, there are 600 problems and solutions in this book. Also included are a few original problems and select power question problems. (The ciphering problems are not included.) Along with that are introductory remarks and illustrative examples in each chapter.

This project started in November 2016 and the first draft of the book was completed in January 2017. This was initially undertaken without the knoweldge of Don and Debbie, as a surprise for them.

Chuck Garner
November 2017

CHAPTER **1**

ALGEBRA

Most of the problems in this section fall into two categories: those which require manipulation of symbols and expressions to reduce or simplify an expression, or those in which a situation is described for which assigning a variable is helpful. Certainly other algebraic techniques are useful, such as solving equations, factoring, and combining like terms. We hope you are familiar enough with those techniques so we can focus on other algebraic things.

Absolute Value. One of those other algebraic things is the idea of *absolute value*. You possibly have some intuitive notion of what absolute value means, but we need a definition. Anything important begins with definitions. Without definitions, we can never know what precisely we are talking about.* Absolute value is no exception.

Definition. The absolute value of x, denoted $|x|$, is defined to be

$$|x| = \begin{cases} x & \text{if } x \text{ is positive or zero} \\ -x & \text{if } x \text{ is negative.} \end{cases}$$

Another way to define the absolute value of x is that $|x|$ is the distance from x to zero along a number line.

For example, by the function-definition of absolute value, we have

$$|-7| = -(-7) = 7.$$

*"Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true." — Bertrand Russell

More intuitively, we note that $|-7|$ is a distance of 7 units from zero on a number line; hence, $|-7| = 7$. This notion of distance helps us solve a variety problems involving absolute value.

Means. Another of the concepts on which we focus on are *means*.

Definition. Suppose a and b are real numbers. The *arithmetic mean* of a and b is $(a+b)/2$ (this is also called simply the average of a and b). The *geometric mean* of a and b is $\sqrt{ab}$. The *harmonic mean* of a and b is $2ab/(a+b)$.

A note about the harmonic mean: this mean is actually the reciprocal of the average of the reciprocals. Indeed,

$$\frac{1}{\dfrac{1/a+1/b}{2}} = \frac{2}{1/a+1/b} = \frac{2}{(a+b)/(ab)} = \frac{2ab}{a+b}.$$

There is more about these means in the chapter on Sequences and Series.

Determinants. In this chapter are also problems on determinants.

Definition. The *determinant* of four real numbers a, b, c, and d is defined to be the quantity $ad - bc$ represented by the symbol

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

In particular, the above expression is called a *2-by-2 determinant*. A *3-by-3 determinant* is found by computing three related 2-by-2 determinants in the following manner.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

This procedure is called the *expansion* along the first row and the associated 2-by-2 determinants are called the *principle minors*.

For example,

$$\begin{aligned} \begin{vmatrix} 2 & 3 & 1 \\ -1 & 2 & 4 \\ 5 & -2 & 3 \end{vmatrix} &= 2\begin{vmatrix} 2 & 4 \\ -2 & 3 \end{vmatrix} - 3\begin{vmatrix} -1 & 4 \\ 5 & 3 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 5 & -2 \end{vmatrix} \\ &= 2[6-(-8)] - 3[-3-20] + [2-10] \\ &= 2[14] - 3[-23] + [-8] \\ &= 28 + 69 - 8 = 89. \end{aligned}$$

Determinants are useful for finding areas and volumes and such. If a square matrix[†] is denoted M, then we denote the determinant of that matrix as $\det M$. An interesting property of matrices and their determinants is the content of the next theorem.

Theorem. *Let M and N be 2-by-2 matrices of real numbers. Then* $\det(MN) = (\det M)(\det N)$.

Proof. Let M and N be the 2-by-2 matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad N = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

where a, b, c, d, p, q, r, and s are real numbers. Then, by definition, $\det M = ad - bc$ and $\det N = ps - qr$. Now, we compute the product MN. This is

$$MN = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}.$$

The determinant of MN is therefore

$$\begin{aligned} \det(MN) &= (ap + br)(cq + ds) - (cp + dr)(aq + bs) \\ &= acpq + adps + bcqr + bdrs - acpq - bcps - adqr - bdrs \\ &= adps + bcqr - bcps - adqr \\ &= (ad - bc)(ps - qr) \\ &= (\det M)(\det N), \end{aligned}$$

which was what we wanted to prove. ❑

As a special case, consider the square of the matrix A: $A \cdot A = A^2$. Then by this theorem, where $M = A$ and $N = A$, we have $\det(A^2) = (\det A)^2$.

Roots of Polynomials. Finally, also collected here are problems concerning roots of polynomials. Most of the problems do not require one to actually find the roots. Most are easily solved if one knows *Viéte's relations*.[‡] The idea behind Viéte's relations is that there is a relationship between the roots of a polynomial and the polynomial's coefficients. This relationship is clear if we consider, for example, the fact that $(x - 3)(x + 2) = x^2 - x - 6$. The roots are clearly -2 and 3. However, upon distributing, we see that 1 is the sum of the

[†] A square matrix is a matrix with the same number of rows as columns.

[‡] Named after 16th century French mathematician, François Viéte (pronounced "vee-et" with the stress on the second syllable).

roots and -6 is the product of the roots. We can express this idea in general: Let $p(x)$ be a polynomial with roots $r_1, r_2, \ldots, r_n$ (not necessarily distinct). Then

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0 \\ &= a_n(x - r_1)(x - r_2)(x - r_3)\cdots(x - r_{n-1})(x - r_n). \end{aligned}$$

The roots are related to the coefficients by the following.

$$\text{Sum of the roots:} \quad -\frac{a_{n-1}}{a_n} = r_1 + r_2 + \cdots + r_n = \sum_{i=1}^{n} r_i$$

$$\text{Sum of the pairwise products:} \quad \frac{a_{n-2}}{a_n} = \sum_{i=1, j=1, i\neq j}^{n} r_i r_j$$

$$\text{Sum of the } k\text{-wise products:} \quad (-1)^k \frac{a_{n-k}}{a_n} = \sum_{i_1 \neq i_2 \neq \cdots \neq i_k}^{n} r_{i_1} r_{i_2} \cdots r_{i_k}$$

$$\text{Product of the roots:} \quad (-1)^n \frac{a_0}{a_n} = r_1 r_2 \cdots r_n = \prod r_i$$

For example, the roots r_1, r_2, r_3, and r_4 of $3x^4 - 5x^3 + 7x^2 - 8x + 1$ must satisfy

$$\begin{aligned} r_1 + r_2 + r_3 + r_4 &= -\frac{-5}{3} = \frac{5}{3} \\ r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 &= \frac{7}{3} \\ r_1r_2r_3 + r_1r_2r_4 + r_2r_3r_4 &= (-1)^3 \frac{-8}{3} = \frac{8}{3} \\ r_1r_2r_3r_4 &= (-1)^4 \frac{1}{3} = \frac{1}{3} \end{aligned}$$

The sum of the squares of the roots, i.e., $r_1^2 + r_2^2 + \cdots + r_n^2$, can also be found in terms of the coefficients. This can be done by generalizing the familiar property that

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc).$$

In our case of n roots, we write

$$\begin{aligned} r_1^2 + r_2^2 + \cdots + r_n^2 &= \left(r_1 + r_2 + \cdots + r_n\right)^2 - 2\left(r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n\right) \\ &= \left(-\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right) \\ &= \left(\frac{a_{n-1}}{a_n}\right)^2 - \frac{2a_{n-2}}{a_n}. \end{aligned}$$

In the example above, $r_1^2 + r_2^2 + r_3^2 + r_4^2 = (5/3)^2 - 2(7/3) = -17/9$.[§]

1.1 Algebra Examples

Example 1.1. (JV 2011, #7)
Twice a number increased by 3 is four times the number decreased by 15. Compute the number.

(A) -9 **(B)** -6 **(C)** -2 **(D)** 6 **(E)** 9

Solution. Here, it useful to assign variables.[¶] Call the number n. Then $2n+3 = 4n - 15$. Solving, we have $18 = 2n$, or $n = 9$. Thus, **E**. ❑

The next example demonstrates how to manipulate an expression involving a variable to answer the question, but without ever determining the value of the variable.

Example 1.2. (Varsity 2005, #25)
If $x - 3 = 5/x$, then find $x^3 - 14x$.

(A) 15 **(B)** 10 **(C)** 2005 **(D)** 101 **(E)** 22

Solution. Working with expressions which have variables in denominators is difficult. So, when possible, clear fractions. With this in mind, we multiply through by x to get $x^2 - 3x = 5$, or $x^2 = 3x + 5$. Now we use this to simplify the expression we wish to find:

$$\begin{aligned} x^3 - 14x &= x\left(x^2 - 14\right) \\ &= x(3x + 5 - 14) \\ &= x(3x - 9) \end{aligned}$$

At this point, it looks like we are stuck. But factoring out a 3 and using the original equation $x - 3 = 5/x$ is the key.

$$\begin{aligned} &= 3x(x - 3) \\ &= 3x\left(\frac{5}{x}\right) \\ &= 3(5) = 15. \end{aligned}$$

[§]The fact that the sum of squares is negative implies that some of the roots are imaginary.
[¶]I’m sure this was a shock.

Note that we used the original relationship $x-3 = 5/x$ twice to help us simplify. In doing so, we did not need to find the actual value of x in order to evaluate the expression.[‖] Thus, **A**. ❑

Example 1.3. (Varsity 2004, #16)
What is the sum of the reciprocals of the roots of the following function?

$$f(x) = x^4 + 2x^3 - 13x^2 - 14x + 24$$

(A) $\frac{7}{12}$ **(B)** $\frac{13}{24}$ **(C)** $-\frac{7}{12}$ **(D)** $-\frac{1}{12}$ **(E)** $-\frac{13}{24}$

Solution. Before we blindly use Viéte's relations, let's investigate just what is being asked. This is a fourth degree polynomial, so there are up to four distinct roots. Call these roots r_1, r_2, r_3, and r_4. Then the sum of the reciprocals is

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{r_2r_3r_4 + r_1r_3r_4 + r_1r_2r_4 + r_1r_2r_3}{r_1r_2r_3r_4}.$$

But the numerator of this fraction is the sum of triple-products (sum of the k-wise products where $k = 3$), and the denominator is the product of the roots. Recall that the coefficient of the x_m term in a polynomial is a_m. Then we may write the sum of the reciprocals in terms of the coefficients:

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{(-1)^3 a_{4-3}/a_4}{(-1)^4 a_0/a_4} = -\frac{a_1}{a_0}.$$

Since we have $a_0 = 24$ and $a_1 = -14$, the sum of the reciprocals of the roots is $-(-14/24) = 7/12$. Thus, **A**. ❑

Example 1.4.
Find all solutions to $\left|2x^3 - 1\right| = 17$.

Solution. We want to determine all values of x such that $2x^3 - 1$ is a distance of 17 from zero. Since both 17 and -17 are 17 units from zero, this implies that either $2x^3 - 1$ is equal to 17 or equal to -17. Hence, we have the two equations

$$2x^3 - 1 = 17 \qquad \text{and} \qquad 2x^3 - 1 = -17.$$

Solving the first one leads to the equation $2x^3 = 18$ so that $x = \sqrt[3]{9}$. Solving the second one leads to the equation $2x^3 = -16$ so that $x = \sqrt[3]{-8} = -2$. Hence the solutions are -2 and $\sqrt[3]{9}$. ❑

[‖] But I'm sure you could find the actual value of x, couldn't you?

The equation in the above example led to two different equations. This is because there will always be two nonzero numbers on a number line which are equal distances from zero (unless the number in question is actually zero, of course).

Example 1.5. (Varsity 2009, #6)
If $f(x) = |3x-4|$ and $g(x) = |1-2x|$ and if $g(f(x)) = 0$, find a possible value for $12x$.

(A) -18 **(B)** -15 **(C)** -6 **(D)** 3 **(E)** 14

Solution. We form the composition** $g(f(x))$ by taking the expression for f and replacing it for x in the function g. This gives

$$g(f(x)) = |1-2f(x)| = |1-2|3x-4||\,.$$

Setting this equal to 0 gives

$$|1-2|3x-4|| = 0$$

which implies that

$$1-2|3x-4| = 0.$$

Isolating the absolute value on the left side gives us $2|3x-4| = 1$ so that either

$$2(3x-4) = 1 \qquad \text{or} \qquad 2(3x-4) = -1.$$

Solving each of these equations gives us, respectively, $x = 9/6$ and $x = 7/6$. Thus the possible values for $12x$ are $12(9/6) = 2\cdot 9 = 18$ and $12(7/6) = 2\cdot 7 = 14$. The only one listed as an answer choice is 14. Thus, **E**. ❑

Most equations involving absolute value will result in more than one equation, and therefore will result in more than one solution. It is possible that some solutions will be extraneous. Any possible solutions should be checked against the original equation.

Another kind of algebra problem is what we call the *defined operation* problem. By "defined operations" we mean those problems which define some new binary operator in terms of familiar operations. These are algebraic or computational in nature, but they cause some people new to competition mathematics a bit of confusion. Let's clear up any confusion with an example or two, ok?

Example 1.6.
Define the operation $a \star b$ for real numbers a and b as $a \star b = a^2 - b^2$. Compute $17 \star (5 \star 3)$.

**For more on the composition of functions, see the chapter on Functions.

Solution. We first compute $5 \star 3$ as $5 \star 3 = 5^2 - 3^2 = 25 - 9 = 16$. Then $17 \star (5 \star 3) = 17 \star 16 = 17^2 - 16^2 = 289 - 256 = 33$. ❑

Example 1.7.
Define the operation $x \triangle y$ for real numbers x and y as $x \triangle y = x^2 + xy - y^2$. Determine the positive value of x such that $x \triangle 8 = 0$.

Solution. We have $x \triangle 8 = x^2 + 8x - 64$, so we want to solve the equation $x^2 + 8x - 64 = 0$. By the quadratic formula,[††]

$$x = \frac{-8 \pm \sqrt{64 + 4 \cdot 64}}{2} = -4 \pm 4\sqrt{5}.$$

The positive value of x is therefore $-4 + 4\sqrt{5}$. ❑

1.2 Algebra Problems

Problem 1.1. (JV 2009, #5)

Emily has two times as many apples as Daniel. Daniel has one-third as many apples as Matthew. Matthew has two more apples than Sawyer. If Sawyer has 52 apples, find the total number of apples that these students have.

(A) 36 **(B)** 54 **(C)** 56 **(D)** 158 **(E)** 160

Problem 1.2. (JV 2003, #1)

Solve: $6(2x - 3(4 - 2x)) = 4((2x - 3) - (5 - 2x))$.

(A) $-\frac{5}{2}$ **(B)** $\frac{5}{2}$ **(C)** $\frac{5}{4}$ **(D)** $\frac{32}{19}$ **(E)** $\frac{89}{16}$

Problem 1.3. (JV 2012, #21)

If $a \oplus b = a + ab^{-1}$, evaluate $3 \oplus (1 \oplus -2)$.

(A) $2\frac{1}{2}$ **(B)** $2\frac{3}{4}$ **(C)** $3\frac{1}{2}$ **(D)** 4 **(E)** 9

[††] See the chapter on Quadratics for more on the quadratic formula.

Problem 1.4. (JV 2005, #16)

Suppose that # is an operation applied to positive real numbers such that $a\#b = a^{b-1}$. What is 3#(2#3)?

(A) 3 **(B)** 8 **(C)** 9 **(D)** 27 **(E)** 81

Problem 1.5. (JV 2003, #23)

Solve for all values of x: $|x - 6| > 2x + 9$.

(A) $6 < x < 15$ **(B)** $-1 < x < 6$ **(C)** $x < -1$ or $x > 6$ **(D)** $x < -1$ **(E)** $x < -15$

Problem 1.6. (JV 2010, #29)

The zeros of a polynomial function $p(x) = 2x^3 - x^2 - ax - 6$ are -2, r_1, and r_2. Find $r_1 + r_2$.

Problem 1.7. (JV 2008, #2)

Suppose that $a = 1$, $b = 3$, $c = 5$, and $d = 11$. If a becomes 3, b and c do not change, and the average value of a, b, c, and d remains constant, then what must d become?

(A) 11 **(B)** 1 **(C)** 13 **(D)** 10 **(E)** 9

Problem 1.8. (JV 2005, #28)

The average test grade for a class of 30 students was 84. If the 18 girls had an average of 90 for the test, what was the average of the boys' test grades?

Problem 1.9. (Varsity 2007, #7)

What is the sum of all the complex roots of the following polynomial?

$$4x^4 + 3x^3 - 37x^2 + 57x - 12$$

(A) -4 **(B)** -3 **(C)** 12 **(D)** $-\frac{57}{4}$ **(E)** $-\frac{3}{4}$

Problem 1.10. (JV 2003, #7)
Which of the following is a factor of $x^3 + 4x^2 - 17x - 60$?

(A) $x - 5$ **(B)** $x - 3$ **(C)** $x - 2$ **(D)** $x + 3$ **(E)** $x + 4$

Problem 1.11. (JV 2009, #11)
Little Bo Peep is having trouble keeping track of her sheep. This time she hasn't lost them, but she doesn't know how many she has. However, she does know that:

- On December 18, 2006, she had 100 sheep.
- Every March, the number of sheep she has triples.
- Every September, she sells 100 sheep.

Given that today is December 5, 2009, how many sheep does she have today?

(A) 100 **(B)** 1400 **(C)** 1500 **(D)** 5100 **(E)** 5200

Problem 1.12. (JV 2005, #27)
Solve for all real values of p: $|5p + 3| = |2p + 2|$.

Problem 1.13. (Varsity 2004, #6)
Evaluate the determinant $\begin{vmatrix} \sin 60^\circ & \cos 30^\circ \\ \cos 180^\circ & \cos 45^\circ \end{vmatrix}$.

(A) $\dfrac{\sqrt{6} + 2\sqrt{3}}{4}$ **(B)** $\dfrac{\sqrt{6} + \sqrt{3}}{4}$ **(C)** $\dfrac{2\sqrt{3} - \sqrt{6}}{4}$ **(D)** $\dfrac{\sqrt{6} - 2\sqrt{3}}{4}$ **(E)** $\dfrac{\sqrt{6}}{4} + \dfrac{1}{2}$

Problem 1.14. (Varsity 2007, #3)
We define $x \perp y$ to be the expression $3x+2y$. What is the value of $(3 \perp 2)-(2 \perp 3)$?

(A) -6 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 6

Problem 1.15. (JV 2008, #4)

How many distinct solutions does the following equation have?

$$\left|x^2 - 3x + 2\right|^2 + 1 = 0$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Problem 1.16. (Varsity 2004, #11)

Simplify the determinant $\begin{vmatrix} 4 & -5 & x \\ 1 & 3 & 6 \\ 7 & 2x & -2 \end{vmatrix}$.

(A) $2x^2 - 69x + 196$ **(B)** $x^2 - 45x + 122$ **(C)** $-2x^2 + 69x + 244$ **(D)** $2x^2 - 69x - 244$ **(E)** $-x^2 + 45x - 98$

Problem 1.17. (Varsity 2012, #6)

The harmonic mean of two numbers is equal to twice their arithmetic mean minus their geometric mean. The sum of the two numbers is 10. Find their product.

(A) -25 **(B)** -10 **(C)** 5 **(D)** 10 **(E)** 25

Problem 1.18. (JV 2005, #26)

If $p(x) = x^3 - 31x + 30$, then what is the smallest root of $p(x)$?

Problem 1.19. (JV 2010, #28)

Two high school classes took the same exam. One class of 35 students had a mean grade of 70 while the other class of 25 had a mean grade of 85. What is the mean grade for all students in both classes?

Problem 1.20. (JV 2009, #2)

If $r \diamond p = 5p^2 + 2r - 1$, evaluate $\dfrac{2 \diamond 1}{-18 \diamond 3} \diamond \dfrac{3 \diamond 1}{-7 \diamond 2}$.

(A) 7 **(B)** 11 **(C)** 17 **(D)** 19 **(E)** 21

Problem 1.21. (JV 2007, #9)

Which of the answer choices is equal to the following expression?

$$\frac{x^{-2}y^3z^{-1}}{x^{-2}-y^{-3}}$$

(A) $\frac{x^2y^3}{x-z}$ **(B)** 1 **(C)** $\frac{y^3\left(x^2-y^3\right)}{x^2z}$ **(D)** xyz **(E)** $\frac{y^6}{y^3z-x^2z}$

Problem 1.22. (Varsity 2010, #22)

Deborah and Donald had the same grade on the last calculus quiz, and, for both of them, it was their highest quiz score this semester. It brought Deborah's quiz average from 83 to 86 and Donald's from 88 to 90. How many quizzes has each student in the class taken?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Problem 1.23. (JV 2011, #23)

Find the sum of 0 and all the solutions to $|3x+21|-8=-2$.

(A) -14 **(B)** -10 **(C)** 0 **(D)** 7 **(E)** 21

Problem 1.24. (JV 2004, #27)

What is the largest root of $p(x)=x^3-5x^2-2x+24$?

Problem 1.25. (Varsity 2007, #9)

Let x be a positive variable. What is the maximum possible value of the expression below?

$$\frac{1}{1+x}+\frac{1}{1+1/x}$$

(A) 1 **(B)** 2 **(C)** $\sqrt{2}$ **(D)** $\frac{3}{2}$ **(E)** 4

Problem 1.26. (JV 2010, #10)

Let $A=\begin{bmatrix}4 & 6\\ -2 & 1\end{bmatrix}$. What is the determinant of A^{-1}?

(A) $\frac{1}{16}$ **(B)** $\frac{1}{8}$ **(C)** $\frac{3}{16}$ **(D)** $\frac{1}{4}$ **(E)** 16

Problem 1.27. (Varsity 2006, #11)

If $f(x) = |3 - 2x|$ and $g(x) = |x - 5|$, find the sum of the two x-values for which $f(g(x)) = x$.

(A) 0 **(B)** $\frac{28}{3}$ **(C)** 13 **(D)** 15 **(E)** $\frac{46}{3}$

Problem 1.28. (JV 2006, #2)

In simplest form, the expression

$$\frac{\frac{4x}{x-4}}{4x - \frac{4x}{x-4}}$$

is equivalent to:

(A) $\frac{1}{x-5}$ **(B)** $\frac{1}{x-4}$ **(C)** $\frac{1}{x-3}$ **(D)** $\frac{1}{x-2}$ **(E)** $\frac{1}{4-x}$

Problem 1.29. (Varsity 2010, #10)

Let $\begin{bmatrix} 3 & 4 \\ 7 & 1 \end{bmatrix}$. What is the determinant of A^3?

(A) -25 **(B)** -376 **(C)** -1465 **(D)** -15625 **(E)** -29750

Problem 1.30. (Varsity 2012, #25)

Let x and y be integers whose harmonic mean is 8 and $x \leq y$. Find the sum of all possible values of x.

Problem 1.31. (Varsity 2005, #27)

Find the sum of the squares of the reciprocals of the roots of $2x^3 - x^2 - 2x + 1 = 0$.

Problem 1.32. (JV 2011, #13)

If $\frac{14x - 30}{3x^2 - 27} = \frac{a}{x-3} + \frac{b}{x+3}$, then compute $3a - b$.

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

Problem 1.33. (JV 2008, #25)

Suppose that $y > 7/2$ and $y^2 - 7y + 3 = 0$. If $z = y + 2$ and $x = y - 3$, then what is the value of $xz - 6x$?

(A) $3\sqrt{37}$ **(B)** 12 **(C)** 9 **(D)** 37 **(E)** 7

Problem 1.34. (Varsity 2008, #17)

Find the sum of all real numbers x satisfying the equation

$$\left|x^2 + 2008x - 2007\right| = \left|x^2 + 2008x - 2009\right|.$$

(A) 0 **(B)** 1004 **(C)** -1004 **(D)** -2008 **(E)** 2008

Problem 1.35. (JV 2003, #27)

For which value(s) of x will $\begin{vmatrix} 2 & 1 & x \\ 0 & -2 & x \\ -2 & x & 3 \end{vmatrix} = \begin{vmatrix} x & 5 \\ 3 & -6 \end{vmatrix}$?

Problem 1.36. (JV 2012, #18)

Find $3 + 2 + \cfrac{3}{2 + \cfrac{3}{2 + \cfrac{3}{2 + \cdots}}}$.

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Problem 1.37. (Varsity 2012, #20)

Given that r is a root of $x^2 + 3x + 6 = 0$, find $r^3 + 4r^2 + 9r + 8$.

(A) -4 **(B)** -2 **(C)** 0 **(D)** 2 **(E)** 4

Problem 1.38. (JV 2006, #17)

If $a + \frac{1}{a} = 3$, what is the value of $\left|a - \frac{1}{a}\right|$?

(A) $2^{1/2}$ **(B)** $\frac{3}{2}$ **(C)** 2 **(D)** 3 **(E)** $5^{1/2}$

Problem 1.39. (Varsity 2006, #9)
Suppose $P(x)$ is a polynomial with integer coefficients that have no common factors. If two of the roots of $P(x)$ are $3 + 4i$ and $2 - \sqrt{6}$, find the y-intercept of $P(x)$ where $P(x)$ has the least possible degree.

(A) -50 **(B)** -25 **(C)** -10 **(D)** 10 **(E)** 50

Problem 1.40. (Varsity 2003, #22)
The & operator is defined as follows for all positive integers a and b:

$$b \,\&\, a = a \,\&\, b = a \,\&\, (a + b).$$

If $5 \,\&\, 6 = 11$, find $7 \,\&\, 25$.

(A) 32 **(B)** 18 **(C)** 11 **(D)** 2 **(E)** Cannot be determined