Five Weeks to a Five

Preparation for the AP Calculus BC Exam

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Five Weeks to a Five: Preparation for the AP Calculus BC Exam

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Preface

This book is designed to prepare a student for taking the AP Calculus BC Exam.

Another exam prep book? There are so many!

Well, yes, there are quite a few available. But they are written for two purposes. Either they are simply collections of practice exams and advice, or they are for self-study. None of them seem to be written with a classroom teacher and their students in mind.

So you're saying this one is different, huh? How so?

I assume that the teacher using this book has spent the school year teaching calculus to their students, and now it is time to prepare for the exam. Through this book's 12 Lessons, students can focus on one aspect of the AP Calculus curriculum from the viewpoint of how it will be assessed on the AP Exam. This is important: the material is presented in terms of how it will be assessed. For example, a calculus teacher probably spent a few class periods teaching students how to use the information from the first and second derivatives to sketch the graph of a function. This should be taught and is important—however, it has been nearly two decades since any free-response question has asked students to sketch the graph of a function. This is simply no longer assessed. (Other graphical aspects of using the derivative are definitely assessed, but not curve sketching.) So curve-sketching is not included in the Lessons, in the problems, or in the practice exams. This is unlike those other books.

What do you mean unlike those other books? Are they filled with curve-sketching problems?

Well, no, but they are filled with interesting things. I have used many different AP Exam Prep books over my career teaching calculus. Two popular books of practice exams in particular have been my "go-to" resource. Until I actually looked at the problems and compared them to the released AP exams. They are not similar in style, and, in some cases, content. These books include curve-sketching free-response problems, antiquated related rates problems, free-response problems where the parts of the problem are too trivial to appear on the AP Exam, and, in one of them, there is even a free-response problem which is all precalculus, except for one part. And yet, these books are supposed to prepare students for the AP Exam. (I'm not saying these problems are not important, or they do not provide good practice, or there is not a pedagogical rationale for asking students to solve them—but questions which do not reflect what students can expect to see on the actual exam isn't really what I'd call "preparation".)

And this doesn't include the other interesting things I find in these books. I went to a Barnes & Noble looking for other possible AP Calculus prep books when I realized my current "go-to"s were not really adequate. I was stunned to see problems on topics no longer in the AP Calculus AB Course Description, such as work problems, marginal cost, and linear first-order differential equations. I saw many multiple-choice questions with directions such as "Use the graph below to answer problems 25-28" which would *never* be found on the AP exam. I also saw BC questions in an AB practice exam! And people buy these books by the thousands.

Then I realized that for a book which should prepare you for the exam, none of them actually *look like the AP Exam!* For instance, the multiple-choice questions are not spaced apart with room to work and separated by horizontal rules, and in those that do space the problems apart, there is an "answer box" for students to write their choice (which is *not* how the AP Exam works). The free-response questions are not printed within a border with the parts separated by a horizontal rule. The graphs and figures are not professionally designed, but instead are copy-and-paste jobs from whatever software the publishers happen to be using! It isn't hard to use the correct typeface, suitable figures, and appropriate spacing to mimic the layout of the AP Exam! If we are going to prepare students for the AP Exam, then let's really get them prepared by offering something that looks like the exam!!

Wow—You sound a little frustrated. It will be okay!

Sorry. But I get so worked up over the lack of thought and care that poor typesetting and formatting indicates!

Allrighty then...So what else is so unique about this book? (Besides this insane Preface which appears to be a dialogue between the personalities in the author's head.) How many problems are there?

At the end of each of the 12 Lessons are 15 multiple-choice problems and 2 free-response problems. There are also four complete practice exams, each with 45 multiple-choice questions and 6 free-response questions. Thus, there are $12 \cdot 15 + 4 \cdot 45 = 360$ multiple-choice questions and $12 \cdot 2 + 4 \cdot 6 = 48$ free-response questions.

That's a lot of problems! Are there answers in the back...?

No. But there is a solution book available to anyone for an outrageously expensive price. The same solution book is available to teachers who bought at least 10 copies of this book for a ridiculously low price.

How do I get the ridiculously low price?

Let me tell you! Teachers emailing me their Lulu.com shipping confirmation which indicates the purchase of at least 10 copies of this book will be emailed a link to purchase the solutions book at the ridiculously low price (plus shipping). This low price is not searchable and can only be accessed by the link I send. The teacher must email the shipping confirmation (that is *shipping confirmation* not receipt of payment!) from their school's email account and it must be sent to `cgarner@gctm.org`. If the teacher does not feel comfortable sending their personal information as listed on the shipping confirmation, they are free to black out or remove that information, as long as the order number, quantity, title, order date, and Lulu.com logo are visible.

Oh, ok—thanks. So how can I use this book with a class?

Each Lesson is designed to be presented/covered/taught/reviewed in one class period, with homework or classwork assinged from the Lesson. As I teach on a block schedule with 90-minute classes meeting every other day, I assign each Lesson as classwork and portions of 3 of the 4 practice exams as homework. I use the fourth practice exam as classwork. This takes me 14 block days, and since we meet every other day, this takes five weeks.

So that's why the title is *Five Weeks to a Five*! I was wondering what's up with that.

Yes, well, it's the best title I could think of that used "five" which wasn't already taken. Meh.

Anyway, the reason each Lesson has 15 multiple-choice and 2 free-reponse is that by assigning three Lessons, students have done the equivalent of an AP Exam ($3 \cdot 15 = 45$ multiple-choice and $2 \cdot 3 = 6$ free-response). Indeed, 4 of the 12 Lessons are allowed the use of a calculator: Areas (Lesson 5), Applications of Calculus (Lesson 8), Volume and Arc Length (Lesson 9), and Parametric Equations and Vectors (Lesson 11). So really, one could mix-and-match the Lessons to create an AP Exam over only certain topics. Indeed, all one needs to do is choose any two of the non-calculator Lessons and any one of the calculator Lessons, and then one has an appropriate-length exam simulation in the proper calculator-to-non-calculator ratios.

To make an AP Exam-length assignment in the proper calculator-to-non-calculator ratio…

assign the problems in any two of these Lessons…	*and any one of these Lessons.*
Limits and Continuity (Lesson 1)	
Derivatives (Lesson 2)	
Antiderivatives (Lesson 3)	Areas (Lesson 5)
Major Theorems of Calculus (Lesson 4)	Applications of Calculus (Lesson 8)
Graphs of Functions (Lesson 6)	Volume and Arc Length (Lesson 9)
Riemann Sums and Trapezoids (Lesson 7)	Parametric and Vector (Lesson 11)
Differential Equations (Lesson 10)	
Series (Lesson 12)	

For instance, if I want my students to really focus on integration and its meaning, I would choose to put together Lessons 3, 5, and 7, on Antiderivatives, Riemann Sums, and Areas. That's just one example of the $\binom{8}{2}\binom{4}{1} = 28 \cdot 4 = 112$ possible combinations of exam-length assignments or assessments you could make.

The Lessons and the four preactice exams give you a quantity of problems equivalent to *eight* AP Exams.

Showing off your counting skills there, I see? Hey, this is for the BC Exam, so where's all the polar stuff?

It's there! I included the slope of polar curves with Parametric Equations, since the slope of a poar curve is approached parametrically (Lesson 11) and the area in polar curves with Areas (Lesson 5). I just didn't give polar it's own Lesson. I did not include a discussion of the graphs of common polar curves (cardioids, limaçons, roses, etc.) since connecting the name of the curves with its equation rarely shows up on the exam without a graph provided, or without a calculator. Besides, the philosophical viewpoint from which I wrote this book implies that a teacher should have already discussed these curves with the students, and we are just reviewing the calculus at this point. So all the calculus of polar curves is included!

I probably could have done something similar with parametric equations and vectors (I could have included differentiation of those curves with the Lesson on Derivatives, for example). I chose not to do this for two reasons. I wanted to put parametric equations and vectors together to emphasize the similarities of the two representations, and to highlight their differences. That was best achieved by giving these functions their own Lesson.

You've been going on and on about the problems. What about the Lessons themselves? If this book isn't for self-study, and a teacher is already teaching the class with a textbook, then why have "lessons" at all?

The Lessons serve as another viewpoint for the students (and possibly the teacher). I have tried to explain topics and theorems solely from the viewpoint of how they will be assessed. This is not a hard-and-fast rule, but more of a guideline. I've also tried to be concise and get straight to the point, without all the extra fluff; that can be expanded upon by the teacher if need be. Finally, the examples are completey worked out, and include some tips on taking the exam. Oh! I've also tried not to rely on any one kind of calculator by staying away from syntax and by not telling you which buttons to press—the teacher can help you there. Besides, I don't know which kind of calculator you're using!

Well, I don't have anymore questions—Wait! I noticed that after all that talk about how a preparation book should "look like" the AP Exam, the problems at the end of each Lesson are not formatted like the exam at all! You've only formatted the practice exams to be like the AP exam! Hypocrite much?

Now hang on—I thought about formatting *all* the problems like the AP Exam, but that would have added another 50 pages and increased the cost. So I was trying to think about the teachers who would be buying this, trying to keep costs down. So this was a conscious decision *not* to format the Lesson problems like the AP Exam.

Hmmm...I'll let it pass this time.

Thanks. Speaking of thanks, let me take this opportunity to thank my students: without them, my life would be boring indeed. And I'd like to thank the Georgia Association of AP Mathematics Teachers (`www.gaapmt.org`): they are a great bunch of friendly and knowledgable teachers who have also taught me a great deal about teaching calculus. And I'd like to thank my wife, Julie, for basically everything.

That was nice! Ok, I'm ready to do some calculus!

Good! Let's get started!

Chuck Garner
CONYERS, GEORGIA
JANUARY 2019

Lesson 5

Areas

The common interpretation of the definite integral of a function f over the interval $[a, b]$ is that this is the area of the region bounded by the graph of f, the x-axis, the line $x = a$, and the line $x = b$. This is certainly useful but we soon get into a tricky situation: the value of the definite integral of f, if f happens to be *below* the x-axis on the interval is *negative*. There is no such thing as a negative area. So how do we approach the interpretation of negative values of a definite integral?

One way is to give the definite integral context. Suppose the function f represents a rate, measured in some unit of quantity per unit time. Then the definite integral of f represents the accumulation of those quantities. So a negative accumulation is a *removal* of those quantities. Thus, we may interpret a negative value of a definite integral not as "negative area," but as a removal of quantity. (And of course, a positive value as an accumulation of quantity.)

You did read the previous Lesson, didn't you?

The area or accumulation interpretations help to make sense of the following properties of definite integrals.

THEOREM 5.1: Properties of Definite Integrals

Let $f(x)$ be a continuous and bounded function on the interval $[a, c]$, with b in the interval $[a, c]$. Then

1. $\displaystyle\int_c^a f(x)\,dx = -\int_a^c f(x)\,dx,$
2. $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx,$
3. $\displaystyle\int_a^a f(x)\,dx = 0.$

Under an accumulation interpretation, a definite integral of a rate from $x = a$ to $x = c$ is the accumulation of the quantity. Thus, the definite integral of a rate *backwards*, from $x = c$ to $x = a$, must be the *removal* of the quantity. Hence, the negative (opposite) of removal must be the accumulation. This is indicated in Property 1 above.

Property 2 is saying that the sum of two accumulations (or areas) over two adjacent intervals is the accumulation (or area) over the union of the intervals.

I think I'm accumulating a headache.

Property 3 says that there is no accumulation (area) at a single point in time.

Let's see how we use these properties to solve a problem.

Example 5.1: Finding Areas

Let f be a continuous function on the interval $[-2, 9]$ such that

$$\int_{-2}^{3} f(x)\,dx = 11, \qquad \int_{9}^{5} f(x)\,dx = 5, \quad \text{and} \quad \int_{3}^{9} f(x)\,dx = 7.$$

Compute $\displaystyle\int_{5}^{-2} f(x)\,dx$.

Solution. First, we notice that one of the given integrals and the integral we are asked to find have the limits reversed. So we rewrite them. We are given

$$\int_{5}^{9} f(x)\,dx = -\int_{9}^{5} f(x)\,dx = -5,$$

and we are asked to find

$$\int_{-2}^{5} f(x)\,dx = -\int_{5}^{-2} f(x)\,dx.$$

Now two of the given integrals describe the areas over the length of the entire interval $[-2, 9]$. Hence,

$$\int_{-2}^{9} f(x)\,dx = \int_{-2}^{3} f(x)\,dx + \int_{3}^{9} f(x)\,dx = 11 + 7 = 18.$$

Now we split the interval again into the subintervals $[-2, 5]$ and $[5, 9]$. The total area must be 18, and we know the area over one of these subintervals.

$$\begin{aligned} \int_{-2}^{5} f(x)\,dx + \int_{5}^{9} f(x)\,dx &= \int_{-2}^{9} f(x)\,dx \\ -\int_{5}^{-2} f(x)\,dx + (-5) &= 18 \\ -\int_{5}^{-2} f(x)\,dx &= 23 \\ \int_{5}^{-2} f(x)\,dx &= -23 \end{aligned}$$

Therefore $\displaystyle\int_{5}^{-2} f(x)\,dx = -23.$ ∎

Example 5.2: Area Under a Curve

Find the area under the graph of $f(x) = 27 - x^3$ and above the x-axis in the first quadrant.

Solution. The graph of $f(x) = 27 - x^3$ intersects the x-axis when $27 - x^3 = 0$. The solution to this equation is $x = 3$. In the first quadrant, we have

$$\int_0^3 f(x)\,dx = 27x - \frac{1}{4}x^4\Big|_0^3 = 81 - \frac{81}{4} = \frac{243}{4}$$

as the area. ∎

Example 5.3: Area Under a Curve *(Calculator Allowed)*

Find the area of the region bounded by the graph of $g(x) = 9\ln(\arctan(2x)) - x$ and the x-axis.

Solution. Since we are not given an interval or any other hints (like "first quadrant"), we must use the calculator to graph this function. The graph of g intersects the x-axis twice, and we use the calculator to find that the intersections occur at $x = 1.0393844$ and $x = 3.096958$. Hence,

$$\int_{1.0393844}^{3.096958} g(x)\,dx = 0.757$$

is the area of the region. ∎

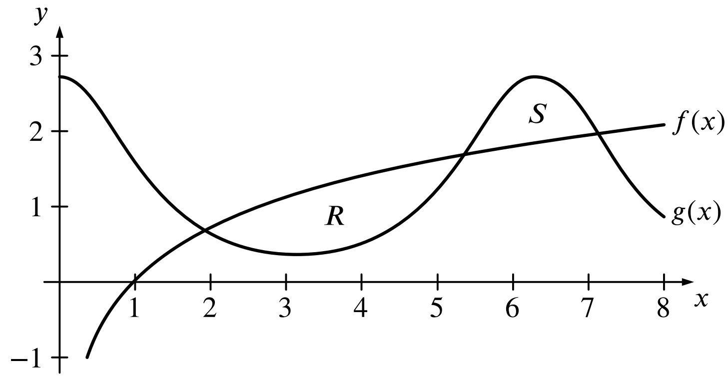
The traditional interpretation of definite integrals as area is a fine one, as long as we know how to deal with "negative area." Area cannot be negative, and this is an important detail in computing the area of a region bounded by the graphs of two functions. The area between two functions can never be negative, so setting up the integral in the proper way is essential. For instance, if a function f is always greater than a function g on an interval, then the area between the curves is the definite integral of $f - g$. On the other hand, if f is always less than g on an interval, we integrate $g - f$. Of course, it could happen that $f \geq g$ for part of the interval and $f \leq g$ for part of the interval. In that case, we need two integrals: one, the integral of $f - g$, for the subinterval where $f \geq g$, and another, the integral of $g - f$, for the subinterval where $f \leq g$.

Those decimals are annoying. Good thing I know how to store them in my calculator!

Example 5.4: Area Between Curves *(Calculators Allowed)*

Find the area of the region(s) enclosed by the graphs of the functions $f(x) = \ln(x)$ and $g(x) = e^{\cos(x)}$.

Solution. That we are told that there could be "region(s)" indicates that we should use the calculator to graph these functions. Their graphs are below.



The graphs enclose two regions, which have been labeled R and S. In region R, we have $f(x) \geq g(x)$ while in region S we have $g(x) \geq f(x)$. Thus, we need to set up two integrals to give us the area, and so we need the intersection points of the graphs. The calculator gives us the intersection points as $x = 1.9699244$, $x = 5.2415078$, and $x = 7.1143419$. Thus, the area of region R is

$$\int_{1.9699244}^{5.2415078} (f(x) - g(x))\, dx = 1.967128158$$

and the area of region S is

$$\int_{5.2415078}^{7.1143419} (g(x) - f(x))\, dx = 1.03971884.$$

Therefore the area enclosed by the curves is the sum of the area of R and the area of S, which is 3.0068.

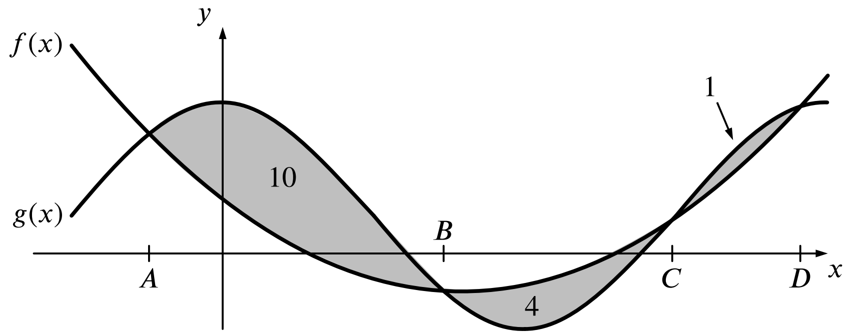
There is another way to find the area. If we compute the integral of $f(x) - g(x)$ to get the area of S, we get the negative of the area. One way around this is to recognize that area is always positive, and simply drop the negative. Another way around this is to integrate the absolute value of $f(x) - g(x)$. In doing this, we can set up and evaluate a single definite integral which gives the area of both regions combined:

$$\int_{1.9699244}^{7.1143419} |f(x) - g(x)|\, dx = 3.0068.$$

The only problem with this is that without a calculator, one would have to know on which subintervals $f(x) - g(x)$ is positive or negative, and this takes you back to the method we used in the first place. ∎

Example 5.5: Area Between Curves

Below are the graphs of two continuous functions, $f(x)$ and $g(x)$. The functions intersect where $x = A$, $x = B$, $x = C$, and $x = D$. The areas of the regions enclosed by the graphs are 10, 4, and 1, as shown.



Compute the following.

1. $\int_A^B (g(x) - f(x))\,dx$
3. $\int_A^C (f(x) - g(x))\,dx$
5. $\int_A^D (f(x) - g(x))\,dx$
2. $\int_C^D (f(x) - g(x))\,dx$
4. $\int_B^D (g(x) - f(x))\,dx$
6. $\int_A^D |f(x) - g(x)|\,dx$

Solution.

1. On the interval $[A, B]$, $g \geq f$, so $\int_A^B (g(x) - f(x))\,dx = 10$.
2. On the interval $[C, D]$, $g \geq f$, so

$$\int_C^D (f(x) - g(x))\,dx = -\int_C^D (g(x) - f(x))\,dx = -1.$$

3. On the interval $[A, C]$, one function is not larger than the other for the whole interval. For $[A, B]$, $g \geq f$, and for $[B, C]$, $f \geq g$. Hence we split the integral into two integrals.

$$\begin{aligned}\int_A^C (f(x) - g(x))\,dx &= \int_A^B (f(x) - g(x))\,dx + \int_B^C (f(x) - g(x))\,dx \\ &= -\int_A^B (g(x) - f(x))\,dx + \int_B^C (f(x) - g(x))\,dx \\ &= -10 + 4 = -6.\end{aligned}$$

4. On the interval $[B, D]$, one function is not larger than the other for the whole interval. For $[B, C]$, $f \geq g$, and for $[C, D]$, $g \geq f$. Hence we split the integral into two integrals.

$$\begin{aligned}\int_B^D (g(x) - f(x))\,dx &= \int_B^C (g(x) - f(x))\,dx + \int_C^D (g(x) - f(x))\,dx \\ &= -\int_B^C (f(x) - g(x))\,dx + \int_C^D (g(x) - f(x))\,dx \\ &= -4 + 1 = -3.\end{aligned}$$

5. Once more, we split the integral.

$$\begin{aligned}\int_A^D (f-g)\,dx &= \int_A^B (f-g)\,dx + \int_B^C (f-g)\,dx + \int_C^D (f-g)\,dx \\ &= -\int_A^B (g-f)\,dx + \int_B^C (f-g)\,dx - \int_C^D (g-f)\,dx \\ &= -10+4-1=-7.\end{aligned}$$

6. With the absolute value of $f(x) - g(x)$, it does not matter which function is larger than the other on any interval. We simply add the areas together to get $10 + 4 + 1 = 15$.

Notice that we called on some of the Properties to evaluate these integrals. ∎

Sometimes considering areas is the only way to evaluate certain kinds of definite integrals, as the next few examples demonstrate.

Example 5.6: Area Without Antiderivatives

Evaluate

$$\int_{-3}^{0} \sqrt{9-x^2}\,dx.$$

Lies! There *is* an antiderivative, but it requires trigonometric substitution which we will *not* get into.

Solution. There is no simple antiderivative of this function, and a calculator answer hides what is really going on. The graph of $y = \sqrt{9-x^2}$ is a semicircle! So we are simply being asked to find the area of a semicircle of radius 3, centered at the origin. In particular, we want the area in the second quadrant bounded by the coordinate axes and the graph — in other words, the area of a quartercircle! The area of a whole circle of radius r is πr^2, so the area of a quartercircle must be $\frac{1}{4}\pi r^2$. Therefore,

$$\int_{-3}^{0} \sqrt{9-x^2}\,dx = \frac{1}{4}\pi \cdot 3^2 = \frac{9\pi}{4}$$

is the answer. No antiderivative necessary! ∎

Example 5.7: Area Under Absolute Value

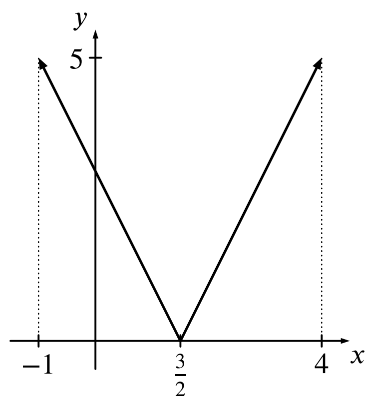
Evaluate

$$\int_{-1}^{4} |2x-3|\,dx.$$

Solution. The function $f(x) = 2x - 3$ is zero when $x = \frac{3}{2}$. This tells us that when $x < \frac{3}{2}$ the function is negative, and when $x > \frac{3}{2}$ the function is positive. Thus, when $x < \frac{3}{2}$, $|2x-3| = 3-2x$, and when $x > \frac{2}{3}$, $|2x-3| = 2x-3$. Hence, we can write and evaluate two definite integrals:

$$\begin{aligned}\int_{-1}^{4} |2x-3|\,dx &= \int_{-1}^{3/2} (3-2x)\,dx + \int_{3/2}^{4} (2x-3)\,dx \\ &= 3x - x^2\Big]_{-1}^{3/2} + x^2 - 3x\Big]_{3/2}^{4} \\ &= \frac{9}{2} - \frac{9}{4} - (-3-1) + 16 - 12 - \left(\frac{9}{4} - \frac{9}{2}\right) \\ &= \frac{9}{4} + 4 + 4 + \frac{9}{4} = \frac{25}{2}.\end{aligned}$$

However, we could also consider the shape of the region formed by the curve over this interval. The graph of $y = |2x-3|$ is below.



Note that this region two right triangles. The triangle on the left has base $\frac{5}{2}$ and height 5, while the triangle on the right also has base $\frac{5}{2}$ and height 5. Therefore, the definite integral is just the sum of the areas of the two right triangles:

$$\int_{-1}^{4} |2x-3|\,dx = \frac{1}{2} \cdot \frac{5}{2} \cdot 5 + \frac{1}{2} \cdot \frac{5}{2} \cdot 5 = \frac{25}{2}.$$

Whicever method we use, we still need to know where $y = |2x-3|$ is zero, and we compute the heights of the triangle by substituting the endpoints of the interval into $y = |2x-3|$. ∎

We may also try to find the area of an unbounded region such as the following example illustrates.

Example 5.8: An Improper Integral

Evaluate

$$\int_{1}^{\infty} \frac{1}{x^3}\,dx.$$

Solution. We are asked to find the area under the curve $y = 1/x^3$ from 1 to *infinity*. This is an unbounded region, since the graph of $y = 1/x^3$ never intersects the x-axis. The way to handle this is to use limits. Let b be a real number. We integrate from $x = 1$

to $x = b$, and then take the limit as $b \to \infty$. We have

$$\begin{aligned}\int_1^\infty \frac{1}{x^3}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^3}\,dx = \lim_{b\to\infty}\int_1^b x^{-3}\,dx \\ &= \lim_{b\to\infty} -\frac{1}{2}x^{-2}\Big]_1^b = \lim_{b\to\infty} -\frac{1}{2x^2}\Big]_1^b \\ &= \lim_{b\to\infty}\left(-\frac{1}{2b^2}+\frac{1}{2}\right) \\ &= 0 + \frac{1}{2} = \frac{1}{2}.\end{aligned}$$

As $b \to \infty$ the fraction $-1/(2b^2) \to 0$, so the area of this unbounded region is simply $1/2$. More accurately, we say that the area *converges* to $1/2$. ∎

Improper integrals such as these may not always converge to a value.

Example 5.9: A Divergent Improper Integral

Evaluate

$$\int_3^\infty \frac{2}{\sqrt{x+1}}\,dx.$$

Solution. Once again, we write this improper integral as the limit, as $b \to \infty$, of the proper integral from $x = 3$ to $x = b$, then evaluate it.

$$\begin{aligned}\int_3^\infty \frac{2}{\sqrt{x+1}}\,dx &= \lim_{b\to\infty}\int_3^b \frac{2}{\sqrt{x+1}}\,dx = \lim_{b\to\infty}\int_3^b 2(x+1)^{-1/2}\,dx \\ &= \lim_{b\to\infty} 4\sqrt{x+1}\Big]_3^b = \lim_{b\to\infty}\left(4\sqrt{b+1}-8\right) = \infty\end{aligned}$$

Hence, this integral never approaches a value, so we say it diverges. ∎

Example 5.10: Integrating Through a Vertical Asymptote

Evaluate

$$\int_1^3 \frac{2}{x^2-2x}\,dx.$$

Solution. This integral, even though neither of the limits of integration is "infinity," is improper. We are asked to integrate over the interval $[1,3]$ but the function has a vertical asymptote at $x = 2$, which is in our interval. So we split this integral into two integrals: one for the interval $[1,2)$ and the other for the interval $(2,3]$. We integrate each of these separately, and then, if they both converge, add the results together.

The first integral is written as a limit, as $b \to 2$ from the left, of the proper integral fro $x = 1$ to $x = b$. To get the antiderivative, we use partial fractions. We have

$$\int_1^2 \frac{2}{x^2-2x}\,dx = \lim_{b\to 2^-}\int_1^b \frac{2}{x^2-2x}\,dx = \lim_{b\to 2^-}\int_1^b \left(\frac{1}{x-2}-\frac{1}{x}\right)\,dx$$
$$= \lim_{b\to 2^-}(\ln|x-2| - \ln|x|)\Big]_1^b = \lim_{b\to 2^-}\ln\left|\frac{x-2}{x}\right|\Bigg]_1^b$$
$$= \lim_{b\to 2^-}\left(\ln\left|\frac{b-2}{b}\right| - \ln|-1|\right)$$
$$= \lim_{b\to 2^-}\ln\left|1-\frac{2}{b}\right| = -\infty.$$

This integral diverges. We no longer need to calculate the integral over $(2,3]$—if the integral over $(2,3]$ converges, the integral over $[1,3]$ still diverges; if the integral over $(2,3]$ diverges, the integral over $[1,3]$ still diverges. Either way, our answer is that the integral diverges. ∎

Finally, since this Lesson is on areas, we look at the area of a polar curve. This is defined in similar terms to a Riemann sum, but instead of using rectangles on infinitely small widths, we use circular sectors of radius r defined by infinitely small angles θ. This notion gives us the following theorem.

The area of a circular sector of radius r and angle θ (in radians) is $\frac{1}{2}r^2\theta$. Adding up all the sectors on infinitely small angles gives us the integral.

THEOREM 5.2: Polar Area

The area A inside a polar curve from angles $\theta = \alpha$ to $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta.$$

Very often, you may not be given the limits of integration. If asked to find the area inside the entire curve, then limits of $\theta = 0$ and $\theta = 2\pi$ are good choices—but these do not apply to all polar curves! One revolution of the circle $r = 2\cos(\theta)$ is determined by $\theta = 0$ to $\theta = \pi$, for instance.

Another instance where you should think about the limits of integration is when you are asked to find the area between polar curves. For this you will need the intersection points.

Example 5.11: Polar Area

Find the area inside the circle $r_1 = 2\sin(\theta)$ that is outside the cardioid $r_2 = 2-2\cos(\theta)$.

Solution. The first thing to do is find the intersection points: we do this by setting the equations of the curves equal and solving. Hence, the solutions to $2\sin(\theta) = 2-2\cos(\theta)$ are $\theta = 0$ and $\theta = \frac{\pi}{2}$. On the interval $[0, \frac{\pi}{2}]$, we have that $2\sin(\theta) \geq 2-2\cos(\theta)$. (We can determine which one is larger by evaluating these curves at any point in the interval and seeing which one gives the larger value.) Hence, we calculate the area of the circle on this interval, and subtract the area of the cardioid on this interval. The integral we calculate is therefore

$$\frac{1}{2}\int_0^{\pi/2} r_1^2\,d\theta - \frac{1}{2}\int_0^{\pi/2} r_2^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left(r_1^2 - r_2^2\right)\,d\theta$$
$$= \frac{1}{2}\int_0^{\pi/2}\left(4\sin^2(\theta) - (2-2\cos(\theta))^2\right)\,d\theta.$$

Now we evaluate this integral to find the area A requested:

$$\begin{aligned}A &= \frac{1}{2}\int_0^{\pi/2}\left(4\sin^2(\theta) - (2-2\cos(\theta))^2\right)\,d\theta \\ &= \frac{1}{2}\int_0^{\pi/2}\left(4\sin^2(\theta) - \left(4 - 8\cos(\theta) + 4\cos^2(\theta)\right)\right)\,d\theta \\ &= \frac{1}{2}\int_0^{\pi/2}\left(4\sin^2(\theta) - 4 + 8\cos(\theta) - 4\cos^2(\theta)\right)\,d\theta\end{aligned}$$

Using the antiderivatives of the squares of sine and cosine (see Table 3.1 on page 35), we get the antiderivative.

$$\begin{aligned}&= \frac{1}{2}\left(2(\theta - \sin(\theta)\cos(\theta)) - 4\theta + 8\sin(\theta) - 2(\theta + \sin(\theta)\cos(\theta))\right)\Big]_0^{\pi/2} \\ &= 4\sin(\theta) - 2\theta - 2\sin(\theta)\cos(\theta)\Big]_0^{\pi/2} \\ &= 4 - \pi.\end{aligned}$$

Obviously if this were a problem on which you could use your calculator, then you could find the intersection points and evaluate the integral. There is no need to have your calculator in polar mode to do either of these. In fact, you only need your calculator in polar mode if you want to graph these polar curves. ∎

Problems for Lesson 5

Thank goodness!

You may use a calculator on these problems.

1. Suppose $\int_{-2}^{2}\left(x^3 + k\right)\,dx = 20$. Find the value of k.

(A) -1024 (B) -5 (C) 5 (D) 1024

2. Determine the area bounded by the graph of $f(x) = \frac{1}{200}\left(e^{x-2} - 50\right)$, the graph of $g(x) = \cos(x)$, the line $x = \frac{9}{2}$, and the y-axis.

(A) 0.079 (B) 0.087 (C) 1.421 (D) 2.754

3. Find the area inside one petal of the rose $r = 3\sin(3\theta)$ but outside the circle $r = 2$.

(A) 0.886 (B) 1.675 (C) 1.771 (D) 3.351

4. $\int_0^{\sqrt{2}} \sqrt{2-x^2}\, dx =$

(A) $\frac{\pi-2}{2}$ (B) $\frac{\pi}{2}$ (C) $\frac{\pi\sqrt{2}}{2}$ (D) 2π

5. $\int_2^{\infty} \frac{4}{x^5}\, dx$ is

(A) $\frac{1}{16}$ (B) $\frac{1}{8}$ (C) $\frac{1}{2}$ (D) divergent

6. If $\int_{-2019}^{2019} e^{x^2}\, dx = K$, then $\int_{-2019}^{0} e^{x^2}\, dx =$

(A) $-2K$ (B) $-\frac{K}{2}$ (C) $\frac{K}{2}$ (D) $2K$

7. Determine the area enclosed by the graphs of $y = x$ and $y = x^2 - 3x + 3$.

(A) $\frac{2}{3}$ (B) $\frac{4}{3}$ (C) 2 (D) $\frac{14}{3}$

8. Suppose f is a continuous function for all x and that

$$\int_1^4 f(x)\, dx = 7, \qquad \int_2^6 f(x)\, dx = 2, \qquad \text{and} \qquad \int_2^4 f(x) = 4.$$

Evaluate $\int_1^6 f(x)\, dx$.

(A) 5 (B) 8 (C) 9 (D) 13

9. The value of $\int_3^{6000} (9^x - 10^x)\, dx + \int_2^{6000} (10^x - 9^x)\, dx$ is

(A) 95.948 (B) 294.918 (C) 685.783 (D) undefined

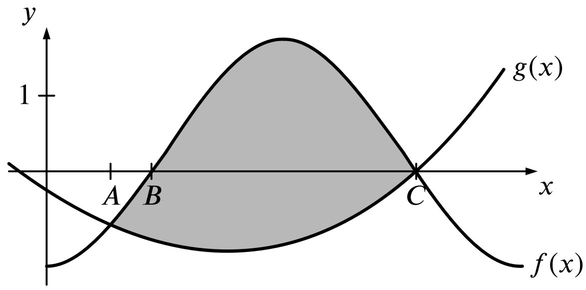
10. Find the area enclosed by the polar curve $r = \sin(\theta)(\cos(3\theta) - 1) + 3$.

(A) 3π (B) 6π (C) $\frac{39\pi}{4}$ (D) $\frac{39\pi}{2}$

11. $\int_1^5 \frac{x}{\left(x^2-9\right)^2}\, dx$ is

(A) $-\frac{3}{32}$ (B) 0 (C) $\frac{3}{32}$ (D) divergent

12. Below are the graphs of two continuous functions f and g. Which of the following expressions can be used to find the area of the shaded region?



(A) $\int_A^C (|f(x)| - |g(x)|)\,dx$ (B) $\int_B^C f(x)\,dx - \int_A^C g(x)\,dx$

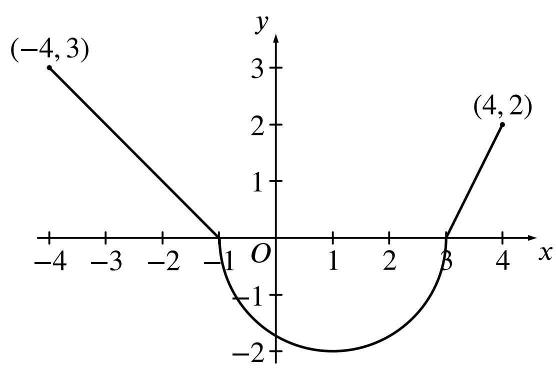
(C) $\int_C^A (g(x) - f(x))\,dx$ (D) $\int_A^B (g(x) - f(x))\,dx + \int_B^C (f(x) - g(x))\,dx$

13. Suppose that $h(x)$ is a continuous function such that $0 \le h(x) \le 8$ for all x where $-2 \le x \le 5$. Determine the greatest possible value of

$$\int_{-2}^{5} h(x)\,dx.$$

(A) 28 (B) 42 (C) 56 (D) 70

14. Below is the graph of a function $g(x)$ which is continuous on the interval $[-4, 4]$. The graph consists of two line segments and a semicircle.



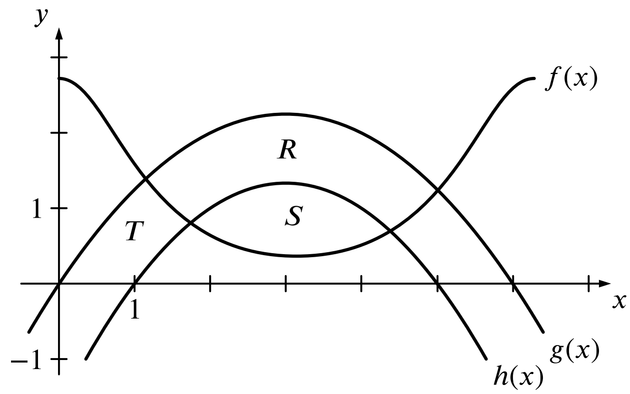
Compute $\int_{-4}^{4} |g(x)|\,dx$.

(A) $\frac{11}{2} - 2\pi$ (B) $7 - 2\pi$ (C) $\frac{11}{2} + 2\pi$ (D) $7 + 2\pi$

15. The region R is bounded by the graphs of $f(x) = 4 - 3x^2$ and $g(x) = 4^{\sin(x)}\cos(x)$. The line $x = k$ divides R into two regions of equal area. Find the value of k.

(A) -0.184 (B) -0.155 (C) -0.103 (D) -0.102

16. The graphs of the functions $f(x) = e^{\cos(x)}$, $g(x) = \frac{1}{4}\left(6x - x^2\right)$, and $h(x) = -\frac{1}{3}\left(x^2 - 6x + 5\right)$ are shown in the figure below.



The region R is bounded by the graphs of f, g, and h. The region S is bounded by the graphs of f and h. The region T is bounded by the graphs of f, g, and h and the x-axis.

(a) Find the area of region S.

(b) Find the area of region R.

(c) Find the area of region T.

(d) The line $x = k$ splits region S into two regions of equal area. Write, but do not evaluate, an equation involving one or more integrals which could be used to find the value of k.

17. Suppose f and g are twice-differentiable functions whose first and second derivatives are both continuous. Values of f and g and their derivatives are given in the table below for certain values of x.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	0	3	2	-6
0	4	$-\frac{1}{2}$	-4	-2
1	3	$-\frac{3}{2}$	-3	4
2	1	-1	9	5

(a) Let $h(x) = \dfrac{f(g(x))}{x^2}$. Find $h'(-1)$.

(b) Let $p(x) = \displaystyle\int_0^{\sin(\pi x/4)} t f(t)\, dt$. Find $p'(2)$.

(c) Evaluate $\displaystyle\int_1^2 \left(x f''(x) + \frac{g'(x)}{g(x)}\right) dx$.

(d) Suppose that $g''(x) \geq f''(x)$ for all x in the interval $[0, 2]$. Find the area between the graphs of $g''(x)$ and $f''(x)$ over the interval $[0, 2]$.