Last-Minute Problems, No. 6

1 Textbook Problems. [3] p.133 #1 and #3.

2 The Incredible Al-Karajî. [4] Abu Bekr ibn Muhammad ibn al-Husayn Al-Karajî (953-1029) was an Islamic mathematician who can be regarded as the first person to loosen algebra from geometrical considerations and instead use operations which can truly be considered algebra. Around the year 1020, Al-Karajî wrote a book on algebra called the *al-Fakhrî*. (The book is named after his patron, Fakhr al-Mulk, the grand vizier of Baghdad at the time.) In it, one is asked to find two rational numbers such that the sum of their cubes is a square. In other words, find rational numbers x, y, and z such that $x^3 + y^3 = z^2$. Al-Karajî takes as the solution

$$x = \frac{n^2}{1+m^3}, \quad y = mx, \quad \text{and} \quad z = nx,$$

where m and n are any rational numbers.

2a) Find x, y, and z when $m = 2$ and $n = 3$. Use these values to verify that $x^3 + y^3$ equals z^2.

2b) What is the relationship between x and y when $m = 1$ and n is even? Why does n have to be even in this case?

2c) Find the exact values (in fractions—no decmials!) of x, y, and z when $m = 1/4$ and $n = 1/8$.

3 The Fantastic Abu Kamīl. [2.5] Abū Kamīl (850-930) was an Islamic mathematician best known for writing a commentary on Al-Khwārizmī's *Algebra*, in addition to writing a number of clever algebra problems. This is one of those clever problems: "The number 50 is divided by a certain other number. If the divisor is increased by 3, then the quotient decreases by 3 and 3/4. What is the divisor?" Solve this problem.

4 Ten Into Two Parts. [10] There is a wide variety of problems found in ancient and medieval mathematics texts, so any similarities stand out easily. One of the peculiar similarities are the problems that we call "ten into two parts" problems. The idea is that 10 is split into two unknown parts, say x and $10 - x$, then some other relationship is defined between the parts that allows one to find the numerical values of the parts. Solve the following problems. The first two are from the works of Al-Khwārizmī (780-850); the next one is from the works of Abū Kamīl (850-930); the last is from the work of one of the first Italian mathematicians, Antonio de' Mazzinghi (1353-1383).

4a) "I have divided 10 into two parts, and having multiplied each part by itself, I have put them together, and have added to them the difference of the two parts previous to their multiplication, and the amount of all this is 54. Find the two parts."

4b) "I have divided 10 into two parts; I multiply the one by 10 and the other by itself, and the products were the same. Find the two parts."

4c) "If one says that 10 is divided into two parts, and one part is multiplied by itself and the other by the root of 8, and subtract the quantity of the product of one part times the root of 8 from the product of the other part multiplied by itself, it gives 40. Find the two parts."

4d) "Divide 10 into two parts such that if one squares the first, subtracts it from 97, and takes its square root, then squares the second, subtracts it from 100, and takes its square root, the sum of the two roots is 17. Find the two parts."

4e) *Bonus for the Bold:* "One says that 10 is divided into two parts, each of which is divided by the other, and when each of the quotients is multiplied by itself and the smaller is subtracted from the larger, then there remains 2. Find the two parts."

$\boxed{5}$ THE MATHEMATICAL POPE. [4] Gerbert (950-1003) was born in France and from an early age, he showed amazing abilities. He was one of the first Christians to study in the Muslim schools in Spain, and may have brought back the Hindu-Arabic numerals to Christian Europe. He is said to have constructed abaci, terrestrial and celestial globes, a clock, and reportedly an organ. Such accomplishments led some to believe that he had sold his soul to the devil. Nevertheless, he rose steadily in the Catholic church and was elected Pope, taking the name Sylvester II, in 999. He was considered a profound scholar, although he contributed very little that was original to mathematics. He did write several textbooks on the subject, however. The following problem is from Gerbert's *Geometry*, and was considered quite difficult at the time; solve it.

5a) Determine the legs of a right triangle whose hypotenuse is 5 and area is 3 and 9/25.

5b) Solve the general case: determine the legs of a right triangle whose hypotenuse is a and has area K.

$\boxed{6}$ FIBONACCI GOT 99 PROBLEMS BUT RABBITS AIN'T ONE. [12.5] Solve the following problems, which appear in the *Liber abaci.*

6a) If A gets from B 7 denarii, then A's sum is fivefold B's; if B gets from A 5 denarii, then B's sum is sevenfold A's. How much has each?

6b) A certain king sent 30 men into his orchard to plant trees. If they could set out 1000 trees in 9 days, in how many days would 36 men set out 4400 trees?

6c) There are two numbers that differ by 5. If the greater is multiplied by $\sqrt{8}$ and the lesser by $\sqrt{10}$, the results are equal. Find the numbers.

6d) A man left to his oldest son one bezant and a seventh of what was left; then, from the remainder, to his next son he left two bezants and a seventh of what was left; then, from the new remainder, to his third son he left three bezants and a seventh of what was left. He continued this way, giving each son one more bezant than the previous son and a seventh of what remained. By this division it developed that the last son received all that was left and all the sons shared equally. How many sons were there and how large was the man's estate? (The man's estate is composed entirely of bezants, which are a type of coin.)

6e) A man entered an orchard through seven gates, and there took a certain number of apples. When he left the orchard he gave the first guard half the apples that he had and one apple more. To the second guard he gave half his apples that he had and one apple more. He did the same to each of the remaining five guards, and left the orchard with one apple. How many apples did he gather in the orchard?

7 Finger Numbers. [2] Finger numbers were widely used for many centuries; from this use, finger processes were developed for some simple computations. One of these processes, by giving the product of two numbers, each between 5 and 10, served to reduce the memory work connected with the multiplication tables. For example, to multiply 7 by 9, raise $7 - 5 = 2$ fingers on one hand and $9 - 5 = 4$ fingers on the other hand. Now add the raised fingers, $2 + 4 = 6$, for the tens digit of the product, and multiply the closed fingers, $3 \times 1 = 3$, for the units digit of the product, giving the result 63. This process is still used by some European peasants. Why does the method gives correct results? *Bonus for the Courageous:* Prove that the method gives correct results.

8 Gelosia. [2] Multiply 80,342 and 7318 by the gelosia method. Make sure you demonstrate the method!

9 Casting Out Nines. [5] The divisibility rule for 9 has been known for hundreds of years: a number is divisible by 9 if the sum of its digits is divisible by 9. This fact was used to determine the following two theorems. The remainder when a number is divided by 9 is called the *excess* for that number.

I. The excess for a sum is equal to sum of the excesses for each addend.

II. The excess for a product of two numbers is equal to the product of the excesses for each factor.

These facts furnish a method for checking arithmetic called *casting out 9s*. For example, to check the calculation $87305 + 93295 = 180600$, we find the excess of each number: $\text{excess}(87305) = 5$, $\text{excess}(93295) = 1$, $\text{excess}(180600) = 6$. Since the excesses add correctly $(5 + 1 = 6)$, the arithmetic must be correct as well.

There is another theorem based on divisibility by 9.

III. If the digits of any integer are rearranged to form a new number, then the difference between the old and new numbers is divisible by 9.

Rearrange 93295 to get 29953. Subtract: $93295 - 29953 = 63342$. $\text{Excess}(63342) = 0$, so 63342 is divisible by 9.

9a) By hand, add 478 and 993, and then, also by hand, multiply 478 and 993. Show that you checked your answers by casting out 9s.

9b) Why does Theorem III work? (An explanation is fine, but a proof is better!)

Theorem III is still used today. It is called the *bookkeeper's check*: if the sums of debit entries and credit entries do not balance, and the difference between the two sums is divisible by 9, then it is quite likely that the error is due to accidentally reversing (transposing) digits when entering a debit or credit.

9c) Explain the following number trick: Someone is asked to think of a number; form a new number by reversing the order of the digits; subtract the smaller from the larger number; multiply the difference by any number whatever; scratch out any single nonzero digit in the product; and announce what is left. The conjurer finds the scratched-out digit by calculating the excess for the announced result and then subtracting this excess from 9.

10 THE MYTH OF GEMATRIA. [5] Since many of the ancient numeral systems were also alphabetical, it was natural to substitute number values for the letters in a name. This led to the pseudo-science called *arithmography*, or *gematria*, which was very popular among the ancients, and was revived during the Middle Ages, when the Hindu-Arabic numerals became prevalant.

Michael Stifel (1487-1567) was an ordained priest who used gematria to predict the world would end on October 18, 1533. This, in itself, was not a punishable offense, but he had persuaded his congregation to give all their wordly goods to him for "safe-keeping" in the hereafter. He was tried for heresy, and turned his obvious mathematical skill to better use by writing books using the new algebraic symbols of Rudolff.

An extreme example of Stifel's gematria is his "proof" that Pope Leo X was the "beast" mentioned in the *Book of Revelation.* From the Latin LEO DECIMVS he retained the letters L, D, C, I, M, and V since these have values in the Roman numeral system. He then added X, both for Leo X and because Leo decimus contains 10 letters, and omitted M because it stands for *mysterium.* A rearrangement of the letters gave DCLXVI, or 666, the "number of the beast."

Some years later, Napier, the inventor of Logarithms, showed (by similar methods) that 666 stands for the Pope of Rome, and a Jesuit priest, Father Bongus, declared that 666 stands for Martin Luther. Bongus' reasoning went as follows. (The Latin alphabet is just like the English, except there is no J and no W, and in upper case, U appears as a V.)

Bongus let the letters A to I represent 1 to 9, K to S represent 10 to 90 (by tens), and T to Z represent 100 to 500 (by hundreds). Then we have, in Latin,

M	A	R	T	I	N	L	V	T	E	R	A
30	1	80	100	9	40	20	200	100	5	80	1

whose sum is 666.

It has been shown that if you express 666 in the numeric letter symbols of the Aramaic language in which the *Book of Revelation* was originally written, 666 spells Nero.

During World War I, gematria was used to show that 666 must be Kaiser Wilhelm, and later it was shown to represent Hitler. By using values assigned to English letters, it was "proven" that, of the three, Roosevelt was a "greater" leader than Churchill or Stalin. More recently, former President Ronald Wilson Reagan has been "proven" to be the beast since each of his three names has six letters.

10a) The word amen, when written in Greek, is $\alpha\mu\eta\nu$. On this evidence, explain why, in certain Christian manuscripts, the number 99 appears at the end of a prayer.

10b) "Prove," by Stifel's method, that each of these represent 666: LVDOVICVS (Ludovicus — King Louis XIV) and SILVESTER SECVNDVS (Silvester Secundus — Pope Sylvester II)

10c) What about your own name? Can you assign values to the English alphabet to prove how "great" (or how beastly!) you are? (Assign values similar to the way in which Bongus assigned values to the Latin alphabet, so your value assignment cannot be random, but must exhibit some logic so it sounds legit!)