Last-Minute Problems, No. 5

1 PTOLEMY'S COROLLARY. [2] Prove the following using Ptolemy's Theorem: If P lies on the arc AB of the circumcircle of an equilateral triangle ABC, then $PC = PA + PB$.

2 APPROXIMATING POLYGONS WITH HERON. [3] Heron (10-75 AD) was the first Greek geometer to be concerned, almost exclusively, with practical problems and procedures (formulas) for finding areas and volumes of various shapes. Heron determined all kinds of area relationships. Although the areas of some regular polygons are relatively easy to find (square and octagon, for instance) Heron approximated the area of many other regular polygons. For example, if we let s be the side length and A_n the area of the regular n-gon, Heron gave

$$A_3 = \frac{13}{30}s^2, \qquad A_5 = \frac{5}{3}s^2, \qquad A_7 = \frac{43}{12}s^2, \qquad A_9 = \frac{51}{8}s^2.$$

2a) Using a calculator and the modern formula

$$A_n = \frac{1}{4}ns^2 \cot\left(\frac{\pi}{n}\right)$$

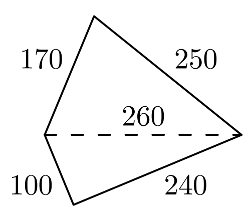
determine the percentage error†† in Heron's approximations.

2b) What approximation for $\sqrt{3}$ is implied by Heron's value for A_3?

2c) It was mentioned above that the areas of a square (A_4) and an octagon (A_8) are relatively wasy to find. So why is there no formula for the area of a hexagon (A_6)?

3 HAVING FUN WITH HERON. [7]

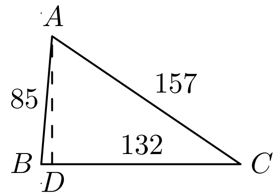
3a) We own a four-sided piece of land, whose area we must determine for tax purposes. Pacing off the sides, we get lengths of 100 yards, 170 yards, 250 yards, and 240 yards (as shown), and then we march along the diagonal and find its length to be 260 yards. How many square yards of property do we own?



3b) An equilateral triangle has each side $4x$ units long, where x is some positive real number. Find its area both by the standard formula ($A = bh/2$) and by Heron's formula and verify that the results agree.

3c) Consider the triangle below.

††Percentage error is |estimated value − actual value| × 100 ÷ actual value.



c1: Find its area via Heron's formula.

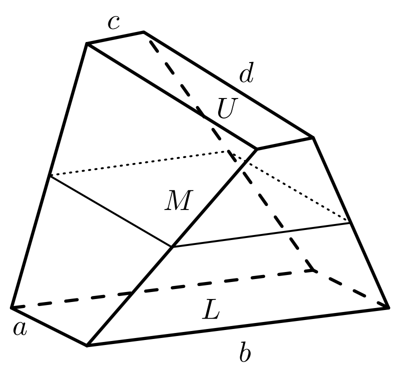
c2: Find its altitude, $\overline{AD}$, using part c1.

c3: What do you notice about $\triangle ABC$ from part c2?

c4: Suppose a circle is inscribed in this triangle. Determine the area that is left if we remove the inscribed circle from $\triangle ABC$. (*Hint:* Use a result from Problem 8e on Last-Minute Problems #2.)

$\boxed{4}$ PRISMATOIDS. [4.5] A polyhedron all of whose vertices lie in two parallel planes is called a *prismatoid*. The section parallel to the bases and midway between them is called the *midsection* of the prismatoid. The figure below shows a prismatoid with rectangular bases. If we denote the areas of the upper base, lower base, and midsection by U, L, and M, we have the modern formula for the volume of any prismatoid of height h:

$$V = \frac{h(U + L + 4M)}{6}.$$



In Book II of Heron's *Metrica*, Heron gives the volume of a prismatoid with rectangular bases. Denote the lower length and width by a and b, and the upper by c and d. Then Heron gives this formula:

$$V = h\left[\frac{(a+c)(b+d)}{4} + \frac{(a-c)(b-d)}{12}\right].$$

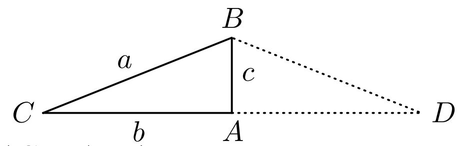
4a) Write U, L, and M in terms of a, b, c, and d.

4b) Show that Heron's formula is equivalent to the modern one.

4c) Show that the Egyptian formula for a truncated pyramid (from Last-Minute Problems #2, Problem 4) is a special case of the prismatoid formula.

$\boxed{5}$ BATTLE OF THE MILLENNIUM: PYTHAGORAS AND HERON SQUARE OFF. [10] In this problem, you will work through two proofs of the Pythagorean Theorem which use Heron's formula for the area of a triangle.

5a) Suppose we have a right triangle BAC with right angle at A with sides a, b, and c. We extend $\overline{CA}$ to D so that $AD = AC = b$ and then we draw $\overline{BD}$, as shown below.



a1: Show that $\triangle BAC \cong \triangle BAD$.

a2: Explain why the semiperimeter of $\triangle DBC$ is $s = a + b$.

a3: Apply Heron's formula to $\triangle DBC$ to deduce that the area of $\triangle DBC$ is equal to $\sqrt{b^2\,(a^2 - b^2)}$.

a4: Apply the simple formula for triangular area ($A = bh/2$) to prove that the area of $\triangle DBC$ is bc.

a5: Finally equate the area expressions from parts a3 and a4, simplify algebraically, and deduce the Pythagorean Theorem.

5b) Beginning with a right triangle of sides a, b, and c, determine its area in two ways: by the standard formula ($A = bh/2$) and by Heron's formula. Equate these and manipulate the resulting, dreadful equation until you have derived the Pythagorean Theorem.

5c) **c1:** Which proof of "Heron implies Pythagoras" do you like better: Problem 5a or 5b?

c2: Do either of these proofs in Problems 5a and 5b provide valid proofs of the Pythagorean Theorem? That is, do they contain circular reasoning?

$\boxed{6}$ Diophantus Got $\varphi\theta$ Problems But Algebra Ain't α. [10] The following problems are all from the *Arithmetica*.

6a) A common technique employed by Diophantus to solve a system of equations is the following. Consider the system

$$\begin{cases} x + y = a \\ x^2 + y^2 = b. \end{cases}$$

Diophantus sets $x = a/2 + z$ and $y = a/2 - z$. (Note that this ensures that $x + y = a$.) He then substitutes these expressions in the second equation; this results in a single quadratic in the new variable z, whose solution is straightforward. Having found z, it is then easy to produce the values of x and y. This technique was employed by many cultures before the Greeks; this leads some to believe that there was transmission of ideas between societies. Solve the following two systems in this manner:

$$\begin{cases} x + y = 20 \\ x^2 + y^2 = 208 \end{cases} \quad \text{and} \quad \begin{cases} x + y = 22 \\ x^2 + y^2 = 274. \end{cases}$$

6b) Diophantus also used a similar technique to the one above for systems of the form

$$\begin{cases} x + y = a \\ x^3 + y^3 = c. \end{cases}$$

In this instance, with $x = a/2 + z$ and $y = a/2 - z$, we have that

$$z = \sqrt{\frac{c - 2(a/2)^3}{3a}}$$

so that x and y are easily found. Use this technique to solve

$$\begin{cases} x + y = 20 \\ x^3 + y^3 = 2240. \end{cases}$$

6c) Solve Problem 17 of *Arithmetica* Book I: Find four numbers, the sum of every arrangement three at a time being 22, 24, 27, and 20.

6d) All we know of Diophantus' personal life is found in the summary of an epitaph given in Pappus' *Collection*, about 350 AD:

> "This tomb holds Diophantus. Ah, how great a marvel! The tomb tells mathematically the measure of his life. God granted him to be a boy for one-sixth of his life, and adding one-twelfth to this, he grew a beard; he was married after another one-seventh, and five years after his marriage God granted him a son—Alas!; after living half his father's life, chill Fate took him. After consoling his grief by mathematics for four years Diophantus ended his life. Tell me the number of years Diophantus lived."

How many years did Diophantus live?

6e) *Bonus for the Brave:*[‡‡] If m is any positive integer and

$$x = m^2, \quad y = (m+1)^2, \quad z = 2(x + y + 1),$$

prove that the three numbers $xy + z$, $yz + x$, and $zx + y$ are all squares.

6f) Use the previous part to solve Problem 13 of *Arithmetica* Book III: Find three numbers such that the product of any two added to the third is a square.

$\boxed{7}$ Linear Problems. [3] The early Hindu mathematicians solved the problem of finding all integer solutions of the linear indeterminate equation $ax + by = c$, where a, b, and c are integers and a and b are relatively prime. In fact, if x_1 and y_1 is a solution, then *all* integer solutions are given by

$$x = x_1 + mb, \qquad y = y_1 - ma$$

where m is an arbitrary integer. For example, all solutions to $9x + 2y = 59$ are found by first finding one solution, usually by inspection or guesswork. One solution to this equation is $x_1 = 5$ and $y_1 = 7$. Then *all* solutions take the form $x = 5 + 2m$ and $y = 7 - 9m$, for all integers m.

[‡‡]There will be various "Bonus" problems throughout these problem sets all semester. The point total for the problem does *not* include the bonus. To be eligible for the bonus, you must complete all other parts of the problem. You only get the bonus points if you answer the bonus problem correctly. The bonus points you receive is equal to the number of students in the class divided by the number of responses (correct or incorrect) to the bonus problem.

7a) Show that $x = 5+2m$ and $y = 7-9m$ solves $9x+2y = 59$ by substituting the expressions for x and y. Then list the solutions for $m = -3, -2, -1, 0, 1, 2, 3$.

7b) Find all positive integer solutions to $7x + 16y = 209$. List the solutions for $m = -3, -2, -1, 0, 1, 2, 3$.

7c) In how many ways can the sum of five dollars be paid in dimes and quarters? (*Hint:* Write an equation in numbers of cents.)

8 The Spectacular Brahmagupta. [6] Brahmagupta was a Hindu astronomer born in 598 AD in northwestern India. Most of his mathematical discoveries were embedded in his great astronomic work *Brāhmasphuṭasiddhānta*, or *Correct Astronomical System of Brahma*. Brahmagupta proved amazing results concerning cyclic quadrilaterals. Consider the following three theorems that he discovered.

I. The area of a cyclic quadrilateral with sides a, b, c, and d is

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where s is the semiperimeter.

II. The diagonals m and n of a cyclic quadrilateral with consecutive sides a, b, c, and d are

$$m = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}} \quad \text{and} \quad n = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}.$$

III. Let x, y, z, X, Y, and Z be positive integers such that $x^2 + y^2 = z^2$ and $X^2 + Y^2 = Z^2$. (In other words they are Pythagorean triples.) If $a = xZ$, $b = zY$, $c = yZ$, and $d = zX$ are consecutive sides of a cyclic quadrilateral, then the diagonals are perpendicular and the diameter of the circle is given by

$$D = \sqrt{\frac{(ad+bc)(ab+cd)}{ac+bd}}.$$

Moreover, the area and diagonals are rational numbers. A cyclic quadrilateral with these properties is called a *Brahmagupta trapezium*.

Find the sides, diagonals, diameter, and area of a Brahmagupta trapezium determined by the Pythagorean triples $(3, 4, 5)$ and $(5, 12, 13)$. Also find the area of the circle.

9 The Broken Bamboo. [3] As a small example of how similar problems occur in different cultures, solve the following two "broken bamboo" problems. The first is from Brahmagupta's *Correct Astronomical System of Brahma* (628); the second is from Yang Hui's updated version of the Chinese work *Nine Chapters*, called *Arithmetic Rules in Nine Sections* (1261).

9a) A bamboo 18 cubits high was broken by the wind. Its top touched the ground 6 cubits from the root. Tell the lengths of the segments of bamboo.

9b) There is a bamboo 10 *ch'ih* high, the upper end of which being broken reaches the ground 3 *ch'ih* from the stem. Find the height of the break.

10 A MIGHTY WORD PROBLEM. [2] Typical of Hindu and much Islamic mathematical writing, the following problem from the works of Mahāvīra (circa 850) illustrates the verbal nature of mathematics before symbolism. Find the smallest possible solution to this indeterminate problem:

> "Into the bright and refreshing outskirts of a forest, which was full of numerous trees with their branches bent down with the weight of flowers and fruits, trees such as jambu trees, lime trees, plantains, areco palms, jack trees, date palms, hintala trees, palmyras, punnāgo trees, and mango trees—outskirts, the various quarters whereof were filled with many sounds of crowds of parrots and cuckoos found near springs containing lotuses with bees roaming about them—into such a forest outskirts a number of weary travelers entered with joy. There were 63 numerically equal heaps of plantain fruit put together and combined with 7 more of those same fruits, and these were equally distributed among 23 travelers so as to have no remainder. You will tell me now the numerical measure of a heap of plantains."