A Look Toward the Heavens

The History of Mathematics, Part 10

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Outline

Ancient Greek Astronomy

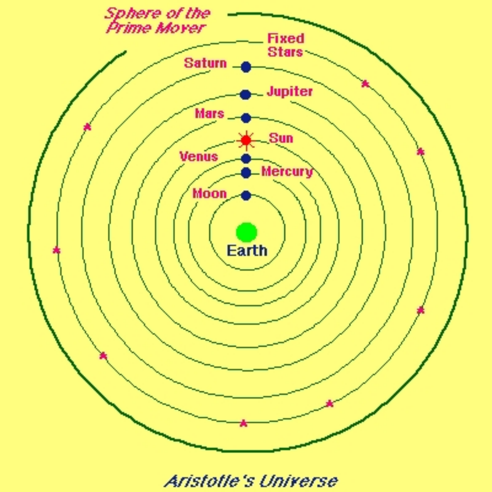
Claudius Ptolemy and the Almagest

Ptolemy's Theorem

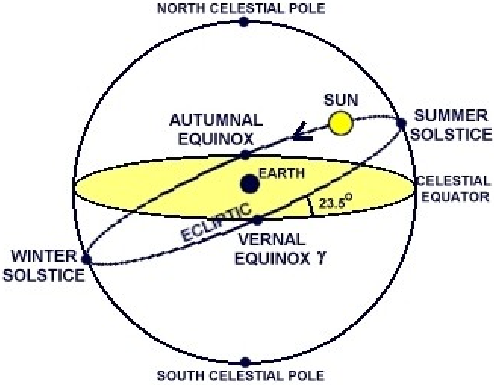
Homework

Outline

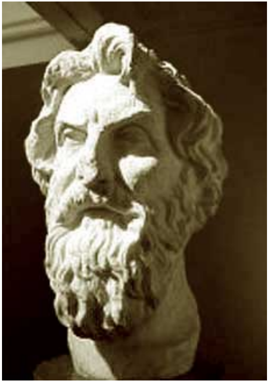
Aristotle


Sphere of the Prime Mover
Fixed Stars
Saturn
Jupiter
Mars
Sun
Venus
Mercury
Moon
Earth
Aristotle's Universe

Aristotle


NORTH CELESTIAL POLE
SUN
AUTUMNAL EQUINOX
SUMMER SOLSTICE
EARTH
CELESTIAL EQUATOR
23.5°
ECLIPTIC
VERNAL EQUINOX ϒ
WINTER SOLSTICE
SOUTH CELESTIAL POLE

Aristarchus



Aristarchus of Samos

310 BC-230 BC

Aristarchus

- Reasoned mathematically that Mercury and Venus revolved around the sun
  So the sun is much bigger than earth…
  So the earth revolves about the sun as well

Aristarchus

- Reasoned mathematically that Mercury and Venus revolved around the sun
  So the sun is much bigger than earth…
  So the earth revolves about the sun as well
- First to give a heliocentric model of the solar system
- Archimedes uses Aristarchus' theory in *The Sand-Reckoner*
- Idea was not popular among the Greeks
- Used the equivalent of

$$\frac{\sin\alpha}{\sin\beta} < \frac{\alpha}{\beta} < \frac{\tan\alpha}{\tan\beta}$$

in his works ($0 < \beta < \alpha < \pi/2$)

Hipparchus

Hipparchus of Rhodes

190 BC–120 BC

“Never deceive a friend.”

Hipparchus

- Careful and precise astronomical observer
- Determined the mean lunar month to within one second
- Accurate calculation of the inclination of the ecliptic
- Discovered procession of equinoxes
- Computed lunar parallax

Hipparchus

- Careful and precise astronomical observer
- Determined the mean lunar month to within one second
- Accurate calculation of the inclination of the ecliptic
- Discovered procession of equinoxes
- Computed lunar parallax
- Catalogued 850 stars
- Advocated use of latitude and longitude
- May have been the first to introduce dividing a circle into 360 parts
- Constructed a table of chords

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Ptolemy

Claudius Ptolemy

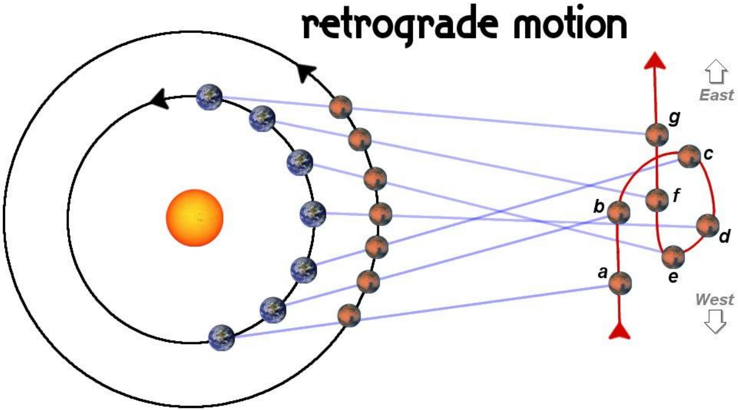
90 AD-170 AD

"When I trace at my pleasure the windings to and fro of the heavenly bodies, I no longer touch the earth with my feet: I stand in the presence of Zeus himself and take my fill of ambrosia, food of the gods."

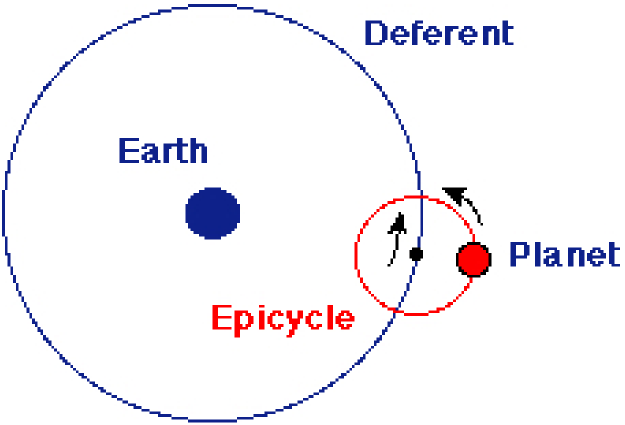
Ptolemy

- Improved Hipparchus’ table of chords
- Wrote the definitive Greek work on astronomy in 13 books
  - *Mathematiki Syntaxis*; trans. *Mathematical Collection*
  - Became known as *Megisti Syntaxis*; trans. *Greatest Collection*
  - Islamic mathematicians traslated it to *al-Magisti*
  - Latin translation of Arabic became *Almagest*

Retrogade Motion



Epicycles



Ptolemy

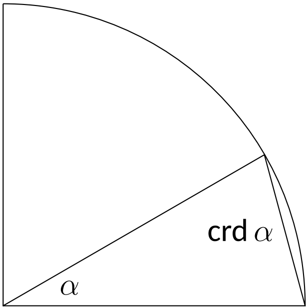
- Was not concerned with *why* only in a geometric model of motion
- Ptolemy calculated where planets and stars would be; to do this, he needed a way to measure chords in circles…
- *Almagest* surpassed all other astronomical works
- Did for astronomy what *Elements* did for geometry

The Table of Chords

The table gives the lengths of chords of all central angles of a circle from $\frac{1}{2}^\circ$ to 180° in half-degree intervals

- Radius of the circle is divided into 60 equal parts
- Chord lengths expressed in base-60 in terms of a radius-part
- Let's use "crd α" to mean the length of a chord of angle α

The Table of Chords



The Table of Chords

- So Ptolemy has

$$\text{crd}\,36^\circ = 37^p\,4'\,55''$$

which means the chord of angle 36° is equal to

  - 37 parts of the 60-part radius, plus
  - $4/60$ of one of these radius parts, plus
  - $55/60$ of one of the 60 parts of the radius part
  - Note similarity to degrees, minutes, seconds

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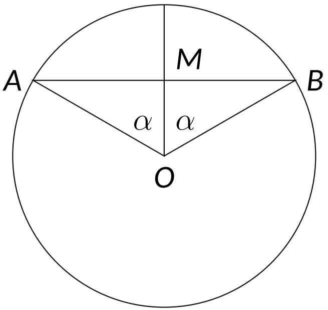
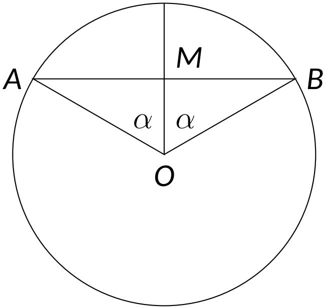
- Let's relate chords to something more modern…

The Table of Chords



$$\sin\alpha = \frac{AM}{OA} = \frac{2\cdot AM}{2\cdot OA} = \frac{AB}{2\cdot 60} = \frac{\text{crd}\,(2\alpha)}{120}$$

The Table of Chords



$$\sin\alpha = \frac{AM}{OA} = \frac{2\cdot AM}{2\cdot OA} = \frac{AB}{2\cdot 60} = \frac{\text{crd}\,(2\alpha)}{120}$$

So the Table of Chords gives sines in quarter-degree intervals from 0° to 90°.

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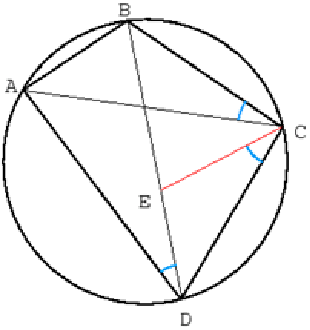
Ptolemy's Theorem

This geometrical gem is not found in the *Elements*

Theorem

In a cyclic quadrilateral the product of the diagonals is equal to the sum of the products of the two pairs of opposite sides.

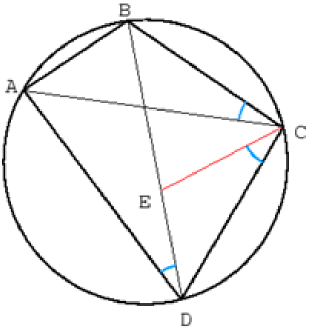
Proof of Ptolemy's Theorem



Let E be on $\overline{BD}$ such that $\angle BCE = \angle ACD$. Then $\triangle BCE \sim \triangle ACD$, so that

$AD/AC = BE/BC$, whence
$AD \cdot BC = BE \cdot AC$.

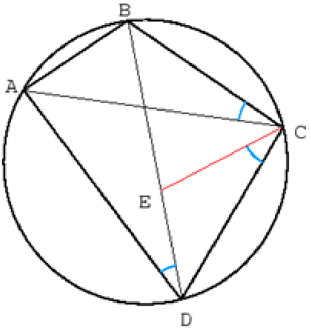
Proof of Ptolemy's Theorem



Let E be on $\overline{BD}$ such that $\angle BCE = \angle ACD$. Then $\triangle BCE \sim \triangle ACD$, so that

$AD/AC = BE/BC$, whence
$AD \cdot BC = BE \cdot AC$.
From similar triangles $\triangle ABC$ and $\triangle DEC$, we have
$AB/AC = DE/DC$, whence
$AB \cdot DC = DE \cdot AC$.

Proof of Ptolemy's Theorem



Let E be on $\overline{BD}$ such that $\angle BCE = \angle ACD$. Then $\triangle BCE \sim \triangle ACD$, so that

$AD/AC = BE/BC$, whence $AD \cdot BC = BE \cdot AC$. From similar triangles $\triangle ABC$ and $\triangle DEC$, we have $AB/AC = DE/DC$, whence $AB \cdot DC = DE \cdot AC$. Therefore

$$\begin{aligned} AD \cdot BC + AB \cdot DC &= BE \cdot AC + DE \cdot AC \\ &= AC(BE + DE) \\ &= AC \cdot BD. \end{aligned}$$

Ptolemy's Corollaries

Corollary (1)

If a and b are the chords of two arcs of a circle of unit radius, then

$$s = \frac{a}{2}\sqrt{4 - b^2} + \frac{b}{2}\sqrt{4 - a^2}$$

is the chord of the sum of the two arcs.

Ptolemy's Corollaries

Corollary (2)

If $a \geq b$ are the chords of two arcs of a circle of unit radius, then

$$d = \frac{a}{2}\sqrt{4 - b^2} - \frac{b}{2}\sqrt{4 - a^2}$$

is the chord of the difference of the two arcs.

Ptolemy's Corollaries

Corollary (3)

If t is the chord of a minor arc of a circle of unit radius, then

$$h = \sqrt{2 - \sqrt{4 - t^2}}$$

is the chord of half the arc.

Calculating the Chord of 1 Degree

- From "Incommensurables" an isosceles triangle with equal sides 1 and vertex angle 36° has base equal to the golden ratio 0.6180
- Hence, in a circle of unit radius,

$$\operatorname{crd} 36^\circ = 60 \times 0.6180 = 37.082.$$

- In a circle of unit radius, $\operatorname{crd} 60^\circ = 1$, so by Corollary 2:

$$\operatorname{crd} 24^\circ = \operatorname{crd}\left(60^\circ - 36^\circ\right) = 22.918.$$

Calculating the Chord of 1 Degree

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Using Corollary 3, we may successively find chords of 12°, 6°, 3°, $\frac{3}{2}^\circ$, and $\frac{3}{4}^\circ$, obtaining

$$\operatorname{crd} \tfrac{3}{2}^\circ = 1.5708 \quad \text{and} \quad \operatorname{crd} \tfrac{3}{4}^\circ = 0.7854.$$

Calculating the Chord of 1 Degree

By the relation $\frac{\operatorname{crd}\alpha}{\operatorname{crd}\beta} < \frac{\alpha}{\beta}$, for $0^\circ < \beta < \alpha < 90^\circ$, we have

$$\frac{\operatorname{crd} 1^\circ}{\operatorname{crd}\,(3/4)^\circ} < \frac{1}{3/4} = \frac{4}{3},$$

or

$$\operatorname{crd} 1^\circ < \frac{4}{3}\operatorname{crd}\left(\frac{3}{4}\right)^\circ = \frac{4}{3} \times 0.7854 = 1.0472.$$

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Also,

$$\frac{\text{crd}\,(3/2)^\circ}{\text{crd}\,1^\circ} < \frac{3/2}{1} = \frac{3}{2},$$

or

$$\text{crd}\,1^\circ > \frac{2}{3}\text{crd}\,\left(\frac{3}{2}\right)^\circ = \frac{2}{3} \times 1.5708 = 1.0472.$$

Hence, $\text{crd}\,1^\circ = 1.0472$.

The *Almagest*

Ninth century copy, in Greek, from the Vatican Library – oldest known copy of the *Almagest*

Ptolemy's Table of Chords

Ptolemy's Table of Chords										
Arc (°)	$Chord_{60}$			$Chord_{10}$	$Sixtieths_{60}$				$Sixtieths_{10}$
0.5	0	31	25	0.523611	0	1	2	50	0.017226
1.0	1	2	50	1.047222	0	1	2	50	0.017226
1.5	1	34	15	1.570833	0	1	2	50	0.017226
2.0	2	5	40	2.094444	0	1	2	50	0.017226
2.5	2	37	4	2.617778	0	1	2	48	0.017226
3.0	3	8	28	3.141111	0	1	2	48	0.017226
3.5	3	39	52	3.664444	0	1	2	48	0.017226
4.0	4	11	16	4.187778	0	1	2	47	0.017226
4.5	4	42	40	4.711111	0	1	2	47	0.017226
5.0	5	14	4	5.234444	0	1	2	46	0.017226
5.5	5	45	27	5.757500	0	1	2	45	0.017226
6.0	6	16	49	6.280278	0	1	2	44	0.017226
6.5	6	48	11	6.803056	0	1	2	43	0.017226

Outline

Homework

- A general survey of Greek mathematics; *Math Through the Ages*, pages 15–24

Next: And Now, Everything Changes